SOLVING OF EQUATIONS AND EIGENVALUE PROBLEMS

Algebraic & Transcendental equations:

The equations of the form $f(x)=0$ where $f(x)$ is purely a polynomial in x is called an algebraic equation. If $f(x)$ involves trigonometrical, logarithmic or exponential terms in it, then it is called transcendental equation.

E.g: ① $x^3-2x-5=0$
② $x^3=6x-4$ } Algebraic equations

③ $e^x=2x+1$
④ $x\log x-1.2=0$
⑤ $3x-\cos x-1=0$ } Transcendental equations

Newton - Raphson method: ((or) Newton's method)

(AU) Condition for convergence : $|f(x)f''(x)| < [f'(x)]^2$

(AU) Order of convergence: Quadratic & is of order 2.

Formula: $x_{n+1}=x_n-\dfrac{f(x_n)}{f'(x_n)}$, $n=0,1,2,\dots$

(AU) ① Find the real root of $xe^x-2=0$ correct to 3 decimal places using Newton-Raphson method.

Sol: Let $f(x)=xe^x-2$

$f'(x)=xe^x+e^x.1$

$f'(x)=xe^x+e^x$

$f(0)=-2 \rightarrow -ve$

$f(1)=0.718 \rightarrow +ve$

$u=x$	$v=e^x$
$u'=1$	$v'=e^x$
$d(uv)=uv'+vu'$	

$\therefore$ A root lies between 0 & 1.

$|f(0)| > |f(1)|$

$|f(0)|=2$
$|f(1)|=0.718$

Hence the root is nearer to 1.

Let $x_0=1$

Formula: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, $n=0,1,2,\ldots$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{f(1)}{f'(1)} = 0.868$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.853$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.853$$

Here $x_2 = x_3 = 0.853$

Hence the required root is 0.853 (correct to 3 decimal places).

(H.W) (AU) ① Find the root of the equation $e^x = 2x+1$, correct to 4 places of decimals, using Newton-Raphson method.

(AU) ② Find the root of $4x - e^x = 0$ that lies between 2 & 3 by Newton-Raphson method. [Ans: 2.1533]

(AU) ② Find the real +ve root of $3x - \cos x - 1 = 0$ by Newton's method correct to 4 decimal places.

Sol: Let $f(x) = 3x - \cos x - 1$

$f'(x) = 3 + \sin x$

$f(0) = -2 \rightarrow -ve$

$f(1) = 1.4597 \rightarrow +ve$

$$\boxed{\begin{array}{l} |f(0)| = 2 \\ |f(1)| = 1.4597 \end{array}}$$

$\therefore$ A root lies between 0 & 1.

$$|f(0)| > |f(1)|$$

Hence the root is nearer to 1.

Let $x_0 = 1$.

Formula: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, $n=0,1,2,\ldots$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.62$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.6071$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.6071$$

Here $x_2 = x_3 = 0.6071$

Hence the required root is 0.6071 (correct to 4 decimal places).

AU ③ Using Newton – Raphson method find the real root of $f(x) = 3x + \sin x - e^{x} = 0$ by choosing initial approximation $x_0 = 0.5$.

Sol: Given $f(x) = 3x + \sin x - e^{x}$ $\qquad f'(x) = 3 + \cos x - e^{x}$

$f(0) = -1 \rightarrow -ve$

$f(1) = 1.1232 \rightarrow +ve$

∴ A root lies between 0 & 1.

Given $x_0 = 0.5$

Formula: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, $n = 0, 1, 2, \ldots$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.3516$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.3604$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.3604$$

Here $x_2 = x_3 = 0.3604$

Hence the required root is 0.3604 (correct to 4 decimal places).

H.W ① Find the +ve root of $f(x) = 2x^3 - 3x - 6 = 0$ by Newton – Raphson method correct to 5 decimal places. [Ans: 1.78377]

② Find the real root of $x = \cos x$ using Newton's method. [Ans: 0.7391]

AU ④ Find the Newton – Raphson formula to find $\sqrt[3]{N}$, where N is a +ve integer.

Sol: Let $x = \sqrt[3]{N} = N^{1/3}$ $\Rightarrow x^3 = N \Rightarrow x^3 - N = 0$

Let $f(x) = x^3 - N$ $\qquad f'(x) = 3x^2$

Formula: $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$, $n = 0, 1, 2, \ldots$

$$x_{n+1} = x_n - \frac{x_n^3 - N}{3x_n^2} = \frac{3x_n^3 - x_n^3 + N}{3x_n^2} = \frac{2x_n^3 + N}{3x_n^2}$$

$$\therefore x_{n+1} = \frac{2x_n^3 + N}{3x_n^2}$$

(AU) ⑤ Find an iterative formula to find the number $\frac{1}{N}$.

Sol: Let $x = \frac{1}{N} \Rightarrow N = \frac{1}{x} \Rightarrow \frac{1}{x} - N = 0$

Let $f(x) = \frac{1}{x} - N$ $\qquad f'(x) = \frac{-1}{x^2}$

Formula: $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$, $n = 0, 1, 2, \ldots$

$$\boxed{\begin{aligned} &\frac{d}{dx}(x^n) = nx^{n-1} \\ &\frac{d}{dx}(x^{-1}) = -1x^{-1-1} = -1x^{-2} \\ &\qquad = \frac{-1}{x^2} \end{aligned}}$$

$$x_{n+1} = x_n - \frac{1/x_n - N}{-1/x_n^2} = x_n - \frac{\left(\frac{1 - x_n N}{x_n}\right)}{-1/x_n^2}$$

$$= x_n + \left(x_n^2 \times \frac{1 - x_n N}{x_n}\right) = x_n + x_n(1 - x_n N)$$

$$= x_n(1 + 1 - x_n N) = x_n(2 - x_n N)$$

$$\therefore x_{n+1} = x_n(2 - x_n N)$$

Fixed point iteration method:

(AU) **Condition for convergence:** $|\phi'(x)| \leq 1$

Order of convergence: 1

(AU) ⑥ Find the smallest +ve root of $x^3 - 2x - 5 = 0$ by the fixed point iteration method, correct to 3 decimal places.

Sol: Let $f(x) = x^3 - 2x - 5$

$f(0) = -5 \rightarrow -ve$

$f(1) = -6 \rightarrow -ve$

$f(2) = -1 \rightarrow -ve$

$f(3) = 16 \rightarrow +ve$

$\therefore$ A root lies between 2 & 3.

$|f(2)| < |f(3)|$

Hence the root is nearer to 2.

Let $x_0 = 2$

Given $x^3-2x-5=0 \Rightarrow x^3=2x+5 \Rightarrow x=(2x+5)^{1/3}=\phi(x)$ (Say)

$\phi'(x)=\frac{1}{3}(2x+5)^{1/3-1}\cdot 2 = \frac{2}{3}(2x+5)^{-2/3} = \frac{2}{3(2x+5)^{2/3}}$

$|\phi'(x)| = \frac{2}{3(2x+5)^{2/3}}$

$|\phi'(2)| = 0.154 < 1$

$|\phi'(3)| = 0.135 < 1$

Here $\phi(x)=(2x+5)^{1/3}$ & $x_0=2$

$x_1=(2x_0+5)^{1/3}=2.08$

$x_2=(2x_1+5)^{1/3}=2.092$

$x_3=(2x_2+5)^{1/3}=2.094$

$x_4=(2x_3+5)^{1/3}=2.094$

Here $x_3=x_4=2.094$

Hence the required root is 2.094 (correct to 3 decimal places)

(AU) ⑦ Find the real root of $\cos x-2x+3=0$ method correct to 3 decimal places using iteration method.

Sol: Let $f(x)=\cos x-2x+3$

$f(0)=4 \rightarrow +ve$

$f(1)=1.54 \rightarrow +ve$

$f(2)=-1.416 \rightarrow -ve$

$\therefore$ A root lies between 1 & 2.

$|f(1)| > |f(2)|$

Hence the root is nearer to 2.

Let $x_0=2$

Given $\cos x-2x+3=0 \Rightarrow 2x=\cos x+3 \Rightarrow x=\frac{\cos x+3}{2}=\phi(x)$ (Say)

$\phi'(x)=-\frac{\sin x}{2}$

$|\phi'(x)|=\frac{\sin x}{2}$

$|\phi'(1)|=\frac{\sin 1}{2}=0.421<1$; $|\phi'(2)|=\frac{\sin 2}{2}=0.455<1$

Here $\phi(x) = \frac{\cos x + 3}{2}$ & $x_0 = 2$

$x_1 = \frac{\cos x_0 + 3}{2} = 1.2919$

$x_2 = \frac{\cos x_1 + 3}{2} = 1.6376$

$x_3 = \frac{\cos x_2 + 3}{2} = 1.4666$

$x_4 = \frac{\cos x_3 + 3}{2} = 1.552$

$x_5 = \frac{\cos x_4 + 3}{2} = 1.5094$

$x_6 = \frac{\cos x_5 + 3}{2} = 1.5307$

$x_7 = \frac{\cos x_6 + 3}{2} = 1.52$

$x_8 = \frac{\cos x_7 + 3}{2} = 1.5254$

$x_9 = \frac{\cos x_8 + 3}{2} = 1.5227$

$x_{10} = \frac{\cos x_9 + 3}{2} = 1.524$

$x_{11} = \frac{\cos x_{10} + 3}{2} = 1.5234$

$x_{12} = \frac{\cos x_{11} + 3}{2} = 1.5237$

$x_{13} = \frac{\cos x_{12} + 3}{2} = 1.5235$

$x_{14} = \frac{\cos x_{13} + 3}{2} = 1.5236$

$x_{15} = \frac{\cos x_{14} + 3}{2} = 1.5236$

Here $x_{14} = x_{15} = 1.5236$

Hence the required root is 1.524 (correct to 3 decimal places).

(AU) ⑧ Compare Gauss elimination method & Gauss-Jordan method for solving a linear system.

Sol:

Gauss elimination method	Gauss-Jordan method
① Direct method ② Coefficient matrix is transformed into upper triangular matrix ③ Back substitution is used	① Direct method ② Coefficient matrix is transformed into diagonal matrix. ③ No need of back substitution.

(H.W) ① Find the real +ve root of $3x - \cos x - 1 = 0$ by fixed point iteration method correct to 4 decimal places.

② Find the root of $4x - e^x = 0$ by fixed point iteration method correct to 4 decimal places.

(AU) ⑨ Solve the system of equations by Gauss-elimination method:

$3x+4y+5z=18$, $2x-y+8z=13$, $5x-2y+7z=20$.

Sol: $[A,B] = \left[\begin{array}{ccc|c} 3 & 4 & 5 & 18 \\ 2 & -1 & 8 & 13 \\ 5 & -2 & 7 & 20 \end{array}\right]$ $R_2 \to 3R_2 - 2R_1$
$R_3 \to 3R_3 - 5R_1$

$$\sim \left[\begin{array}{ccc|c} 3 & 4 & 5 & 18 \\ 0 & -11 & 14 & 3 \\ 0 & -26 & -4 & -30 \end{array}\right] \quad R_3 \to R_3/_{-2}$$

$$\sim \left[\begin{array}{ccc|c} 3 & 4 & 5 & 18 \\ 0 & -11 & 14 & 3 \\ 0 & 13 & 2 & 15 \end{array}\right] \quad R_3 \to 11R_3 + 13R_2$$

$$\sim \left[\begin{array}{ccc|c} 3 & 4 & 5 & 18 \\ 0 & -11 & 14 & 3 \\ 0 & 0 & 204 & 204 \end{array}\right]$$

Hence $3x+4y+5z=18$ — ①

$-11y+14z=3$ — ②

$204z=204 \Rightarrow \boxed{z=1}$ — ③

Substituting ③ in ②, $-11y+14=3 \Rightarrow -11y=-11 \Rightarrow \boxed{y=1}$ — ④

Substituting ③ & ④ in ①, $3x+4+5=18 \Rightarrow 3x=9 \Rightarrow \boxed{x=3}$

Hence the solution is $x=3, y=1, z=1$.

(H.W) (AU) ① Solve by Gauss elimination method: $10x+y=7$ & $x-10y=31$.

(AU) ② Solve the following system of equations by using Gauss-elimination method: $3x-y+2z=12$, $x+2y+3z=11$, $2x-2y-z=2$.

(AU) ⑩ Solve the system of equations by Gauss-Jordan method:

$3x-y+2z=12$, $x+2y+3z=11$, $2x-2y-z=2$.

Sol: $[A,B] = \left[\begin{array}{ccc|c} 3 & -1 & 2 & 12 \\ 1 & 2 & 3 & 11 \\ 2 & -2 & -1 & 2 \end{array}\right]$ $R_2 \to 3R_2 - R_1$
$R_3 \to 3R_3 - 2R_1$

$$\sim \left[\begin{array}{ccc|c} 3 & -1 & 2 & 12 \\ 0 & 7 & 7 & 21 \\ 0 & -4 & -7 & -18 \end{array}\right] \quad R_2 \to R_2/7$$

$$\sim \left[\begin{array}{ccc|c} 3 & -1 & 2 & 12 \\ 0 & 1 & 1 & 3 \\ 0 & -4 & -7 & -18 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 + R_2 \\ R_3 \to R_3 + 4R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 3 & 0 & 3 & 15 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -3 & -6 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1/3 \\ R_3 \to R_3/-3 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 - R_3 \\ R_2 \to R_2 - R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Hence $x = 3$, $y = 1$, $z = 2$.

(H.W) (AU) ① Solve the system of equations by Gauss-Jordan method:
$4x + y + 2z = 7$, $2x + 3y - z = -3$, $x - 2y + 2z = 7$.

(AU) ② Solve $3x + 2y = 4$, $2x - 3y = 7$ by using Gauss-Jordan method.

(AU) (11) Find the inverse of the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{pmatrix}$ by Gauss-Jordan method.

Sol: $[A, I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 4 & 3 & -1 & 0 & 1 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -5 & -4 & 1 & 0 \\ 0 & 2 & 0 & -3 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 + R_2 \\ R_3 \to R_3 + 2R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -4 & -3 & 1 & 0 \\ 0 & -1 & -5 & -4 & 1 & 0 \\ 0 & 0 & -10 & -11 & 2 & 1 \end{array}\right] \quad \begin{array}{l} R_1 \to 10R_1 - 4R_3 \\ R_2 \to 2R_2 - R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 10 & 0 & 0 & 14 & 2 & -4 \\ 0 & -2 & 0 & 3 & 0 & -1 \\ 0 & 0 & -10 & -11 & 2 & 1 \end{array}\right] \begin{array}{l} R_1 \to R_1/10 \\ R_2 \to R_2/-2 \\ R_3 \to R_3/-10 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7/5 & 1/5 & -2/5 \\ 0 & 1 & 0 & -3/2 & 0 & 1/2 \\ 0 & 0 & 1 & 11/10 & -1/5 & -1/10 \end{array}\right]$$

Hence $A^{-1} = \begin{bmatrix} 7/5 & 1/5 & -2/5 \\ -3/2 & 0 & 1/2 \\ 11/10 & -1/5 & -1/10 \end{bmatrix}$

H.w

AU ① Find the inverse of $A = \begin{pmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix}$ using Gauss-Jordan method.

Ans: $A^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{pmatrix}$

AU **Diagonal dominance:**

The absolute values of the leading diagonal elements of the coefficient matrix A of the system $AX=B$ are greater than the sum of absolute values of the other coefficients of that row. This condition is called diagonal dominance.

Compare Gauss-Jacobi & Gauss-Seidel methods:

Sol:

Gauss-Jacobi	Gauss-Seidel
① Indirect method ② The rate of convergence is slow.	① Indirect method ② The rate of convergence is roughly twice that of Gauss-Jacobi.
③ Condition for convergence is the coefficient matrix is diagonally dominant.	③ Condition for convergence is the coefficient matrix is diagonally dominant.

(11) Why Gauss-Seidel method is better than Gauss-Jacobi method?

Sol:

The rate of convergence in Gauss-Seidel method is roughly two times than that of Gauss-Jacobi method.

⑫ Solve the following system of equations by using Gauss-Jacobi method: $8x-3y+2z=20$, $4x+11y-z=33$, $6x+3y+12z=35$.

Sol: $8x-3y+2z=20 \Rightarrow x=\frac{1}{8}(20+3y-2z)$

$4x+11y-z=33 \Rightarrow y=\frac{1}{11}(33-4x+z)$

$6x+3y+12z=35 \Rightarrow z=\frac{1}{12}(35-6x-3y)$

Let the initial values be $x^{(0)}=0, y^{(0)}=0, z^{(0)}=0$.

First iteration:

$x^{(1)}=\frac{1}{8}(20+3y^{(0)}-2z^{(0)})=\frac{20}{8}=2.5$

$y^{(1)}=\frac{1}{11}(33-4x^{(0)}+z^{(0)})=\frac{33}{11}=3$

$z^{(1)}=\frac{1}{12}(35-6x^{(0)}-3y^{(0)})=\frac{35}{12}=2.9167$

Second iteration:

$x^{(2)}=\frac{1}{8}(20+3y^{(1)}-2z^{(1)})=2.8958$

$y^{(2)}=\frac{1}{11}(33-4x^{(1)}+z^{(1)})=2.3561$

$z^{(2)}=\frac{1}{12}(35-6x^{(1)}-3y^{(1)})=0.9167$

3rd iteration:

$x^{(3)}=3.1544$

$y^{(3)}=2.0303$

$z^{(3)}=0.8797$

4th iteration:

$x^{(4)}=3.0414$

$y^{(4)}=1.9329$

$z^{(4)}=0.8319$

5th iteration:

$x^{(5)}=3.0169$

$y^{(5)}=1.9697$

$z^{(5)}=0.9127$

6th iteration:

$x^{(6)}=3.0105$, $y^{(6)}=1.9859$, $z^{(6)}=0.9158$

7th iteration:

$x^{(7)} = 3.0158$

$y^{(7)} = 1.9885$

$z^{(7)} = 0.9149$

8th iteration:

$x^{(8)} = 3.017$

$y^{(8)} = 1.9865$

$z^{(8)} = 0.9116$

Hence $x = 3.02$, $y = 1.99$, $z = 0.91$ (correct to 2 decimal places).

(No) ⑬ Solve the system of equations by using Gauss-Seidel method:

$28x + 4y - z = 32$, $x + 3y + 10z = 24$, $2x + 17y + 4z = 35$.

Sol: Given $28x + 4y - z = 32 \Rightarrow x = \frac{1}{28}(32 - 4y + z)$

$2x + 17y + 4z = 35 \Rightarrow y = \frac{1}{17}(35 - 2x - 4z)$

$x + 3y + 10z = 24 \Rightarrow z = \frac{1}{10}(24 - x - 3y)$

Let the initial values be $y^{(0)} = 0$, $z^{(0)} = 0$.

1st iteration:

$$x^{(1)} = \frac{1}{28}\left(32 - 4y^{(0)} + z^{(0)}\right) = \frac{32}{28} = 1.1429$$

$$y^{(1)} = \frac{1}{17}\left(35 - 2x^{(1)} - 4z^{(0)}\right) = 1.9244$$

$$z^{(1)} = \frac{1}{10}\left(24 - x^{(1)} - 3y^{(1)}\right) = 1.7084$$

2nd iteration:

$$x^{(2)} = \frac{1}{28}\left(32 - 4y^{(1)} + z^{(1)}\right) = 0.929$$

$$y^{(2)} = \frac{1}{17}\left(35 - 2x^{(2)} - 4z^{(1)}\right) = 1.5476$$

$$z^{(2)} = \frac{1}{10}\left(24 - x^{(2)} - 3y^{(2)}\right) = 1.8428$$

3rd iteration:

$x^{(3)} = 0.9876$

$y^{(3)} = 1.509$

$z^{(3)} = 1.8485$

4th iteration:

$x^{(4)} = 0.9933$

$y^{(4)} = 1.507$

$z^{(4)} = 1.8486$

5th iteration:

$x^{(5)} = 0.9936$

$y^{(5)} = 1.507$

$z^{(5)} = 1.8485$

6th iteration:

$x^{(6)} = 0.9936$

$y^{(6)} = 1.507$

$z^{(6)} = 1.8485$

Hence $x = 0.9936$, $y = 1.507$, $z = 1.8485$ (correct to 4 decimal places).

Power method:

(AU) ⑭ Using power method find the largest eigenvalue & the corresponding eigenvector of the matrix $\begin{pmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{pmatrix}$ with initial vector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Sol: Let $A = \begin{pmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{pmatrix}$ & $X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

$$AX_1 = \begin{pmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 9 \\ 13 \end{pmatrix} = 13\begin{pmatrix} 0.231 \\ 0.692 \\ 1 \end{pmatrix} = 13X_2$$

$$AX_2 = \begin{pmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{pmatrix}\begin{pmatrix} 0.231 \\ 0.692 \\ 1 \end{pmatrix} = \begin{pmatrix} 1.307 \\ 6.077 \\ 12.537 \end{pmatrix} = 12.537\begin{pmatrix} 0.104 \\ 0.485 \\ 1 \end{pmatrix} = 12.537X_3$$

$$AX_3 = \begin{pmatrix} 0.559 \\ 5.282 \\ 11.836 \end{pmatrix} = 11.836\begin{pmatrix} 0.047 \\ 0.446 \\ 1 \end{pmatrix} = 11.836X_4$$

$$AX_4 = \begin{pmatrix} 0.385 \\ 5.033 \\ 11.737 \end{pmatrix} = 11.737\begin{pmatrix} 0.863 \\ 0.429 \\ 1 \end{pmatrix} = 11.737X_5$$

$$AX_5 = \begin{pmatrix} 1.15 \\ 7.447 \\ 10.853 \end{pmatrix} = 10.853\begin{pmatrix} 0.106 \\ 0.686 \\ 1 \end{pmatrix} = 10.853X_6$$

$$AX_6 = \begin{pmatrix} 1.164 \\ 5.69 \\ 12.638 \end{pmatrix} = 12.638\begin{pmatrix} 0.092 \\ 0.45 \\ 1 \end{pmatrix} = 12.638X_7$$

$$AX_7 = \begin{pmatrix} 0.442 \\ 5.176 \\ 11.708 \end{pmatrix} = 11.708\begin{pmatrix} 0.038 \\ 0.442 \\ 1 \end{pmatrix} = 11.708X_8$$

$$AX_8 = \begin{pmatrix} 0.364 \\ 4.998 \\ 11.73 \end{pmatrix} = 11.73\begin{pmatrix} 0.031 \\ 0.426 \\ 1 \end{pmatrix} = 11.73X_9$$

$$AX_9 = \begin{pmatrix} 0.309 \\ 4.945 \\ 11.673 \end{pmatrix} = 11.673\begin{pmatrix} 0.026 \\ 0.424 \\ 1 \end{pmatrix} = 11.673X_{10}$$

$$AX_{10} = \begin{pmatrix} 0.298 \\ 4.926 \\ 11.67 \end{pmatrix} = 11.67\begin{pmatrix} 0.026 \\ 0.422 \\ 1 \end{pmatrix} = 11.67X_{11}$$

Hence the largest eigenvalue is 11.67 & the corresponding eigenvector is

$$\begin{pmatrix} 0.03 \\ 0.42 \\ 1 \end{pmatrix}.$$

Jacobi method:

⑮ Apply Jacobi process to evaluate the eigenvalues & eigenvectors of the matrix $\begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix}$.

Sol: Here the largest non-diagonal element is $a_{13} = a_{31} = 1$. Also $a_{11} = 5$, $a_{33} = 5$.

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2a_{13}}{a_{11}-a_{33}}\right) = \frac{1}{2}\tan^{-1}\left(\frac{2}{5-5}\right) = \frac{1}{2}\tan^{-1}\infty = \frac{1}{2}\left(\frac{\pi}{2}\right) = \frac{\pi}{4}.$$

$$P = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\pi/4 & 0 & -\sin\pi/4 \\ 0 & 1 & 0 \\ \sin\pi/4 & 0 & \cos\pi/4 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}$$

$$D = P^TAP = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix},$$

$$= \begin{pmatrix} 6 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

Hence the eigenvalues are 6, −2, 4 & the corresponding eigenvectors are $\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$.

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⑯ Apply Jacobi process to evaluate the eigenvalues & eigenvectors of the matrix $\begin{pmatrix} 4 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix}$.

Sol:

Here the largest non-diagonal element is $a_{23}=a_{32}=2$. Also $a_{22}=1$, $a_{33}=1$.

$$\theta=\frac{1}{2}\tan^{-1}\left(\frac{2a_{23}}{a_{22}-a_{33}}\right)=\frac{1}{2}\tan^{-1}\left(\frac{4}{1-1}\right)=\frac{1}{2}\tan^{-1}\infty=\frac{1}{2}\left(\frac{\pi}{2}\right)=\frac{\pi}{4}.$$

$$P=\begin{pmatrix}1&0&0\\0&\cos\theta&-\sin\theta\\0&\sin\theta&\cos\theta\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1/\sqrt{2}&-1/\sqrt{2}\\0&1/\sqrt{2}&1/\sqrt{2}\end{pmatrix}$$

$$D=P^TAP=\begin{pmatrix}1&0&0\\0&1/\sqrt{2}&1/\sqrt{2}\\0&-1/\sqrt{2}&1/\sqrt{2}\end{pmatrix}\begin{pmatrix}4&1&1\\1&1&2\\1&2&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1/\sqrt{2}&-1/\sqrt{2}\\0&1/\sqrt{2}&1/\sqrt{2}\end{pmatrix}$$

$$=\begin{pmatrix}4&\sqrt{2}&0\\\sqrt{2}&3&0\\0&0&-1\end{pmatrix}=A_1\ (\text{Say})$$

In the matrix A_1, the largest non-diagonal element is $a_{12}=a_{21}=\sqrt{2}$. Also $a_{11}=4$, $a_{22}=3$.

$$\theta=\frac{1}{2}\tan^{-1}\left(\frac{2a_{12}}{a_{11}-a_{22}}\right)=\frac{1}{2}\tan^{-1}\left(\frac{2\sqrt{2}}{4-3}\right)=\frac{1}{2}\tan^{-1}(2\sqrt{2})=0.615$$

$$P_1=\begin{pmatrix}\cos\theta&-\sin\theta&0\\\sin\theta&\cos\theta&0\\0&0&1\end{pmatrix}=\begin{pmatrix}0.817&-0.577&0\\0.577&0.817&0\\0&0&1\end{pmatrix}$$

$$D=P_1^TA_1P_1=\begin{pmatrix}0.817&0.577&0\\-0.577&0.817&0\\0&0&1\end{pmatrix}\begin{pmatrix}4&\sqrt{2}&0\\\sqrt{2}&3&0\\0&0&-1\end{pmatrix}\begin{pmatrix}0.817&-0.577&0\\0.577&0.817&0\\0&0&1\end{pmatrix}$$

$$=\begin{pmatrix}5&0&0\\0&2&0\\0&0&-1\end{pmatrix}$$

$$P=PP_1=\begin{pmatrix}1&0&0\\0&1/\sqrt{2}&-1/\sqrt{2}\\0&1/\sqrt{2}&1/\sqrt{2}\end{pmatrix}\begin{pmatrix}0.817&-0.577&0\\0.577&0.817&0\\0&0&1\end{pmatrix}=\begin{pmatrix}0.817&-0.577&0\\0.408&0.578&-0.707\\0.408&0.578&0.707\end{pmatrix}$$

Hence the eigenvalues are $5, 2, -1$ & the corresponding eigenvectors are $\begin{pmatrix}0.817\\0.408\\0.408\end{pmatrix}$, $\begin{pmatrix}-0.577\\0.578\\0.578\end{pmatrix}$ & $\begin{pmatrix}0\\-0.707\\0.707\end{pmatrix}$.

Au ⑰ Find the eigenvalues & eigenvectors of $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix}$ using Jacobi method.

Sol: Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix}$

Here the largest non-diagonal element is $a_{23} = a_{32} = -1$. Also $a_{22} = 3$

$a_{33} = 3$.

$$\theta = \frac{1}{2} \tan^{-1}\left(\frac{2a_{23}}{a_{22} - a_{33}} \right) = \frac{1}{2} \tan^{-1}\left(\frac{-2}{3-3} \right) = \frac{1}{2} \tan^{-1} \infty = \frac{1}{2}\left(\frac{\pi}{2} \right) = \frac{\pi}{4}$$

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$D = P^T A P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

Hence the eigenvalues are 1, 2, 4 & the corresponding eigenvectors are $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$ & $\begin{pmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$.

INTERPOLATION AND APPROXIMATION

Lagrange's interpolation formula:

Let $y=f(x)$ be a funʃ. which takes the values $y_0, y_1, \dots, y_n$ corresponding to $x=x_0, x_1, \dots, x_n$. Then Lagrange's interpolation formula is

$$y=f(x)=\frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)}y_0+\frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)}y_1$$

$$+\cdots+\frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})}y_n.$$

Problems:

① Find $f(3)$ by using Lagrange's formula

x : 0 1 2 5

$f(x)$: 2 3 12 147

Sol: Given $x_0=0$, $x_1=1$, $x_2=2$, $x_3=5$, $y_0=2$, $y_1=3$, $y_2=12$, $y_3=147$.

Lagrange's interpolation formula is

$$y=f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$=\frac{(x-1)(x-2)(x-5)}{(-10)}(2)+\frac{x(x-2)(x-5)}{4}(3)+\frac{x(x-1)(x-5)}{(-6)}(12)$$

$$+\frac{x(x-1)(x-2)}{60}(147)$$

$$\therefore f(3)=\frac{(2)(1)(-2)}{(-10)}(2)+\frac{3(1)(-2)}{4}(3)+\frac{3(2)(-2)}{(-6)}(12)+\frac{3(2)(1)}{60}(147)$$

$$=\frac{4}{5}-\frac{9}{2}+24+\frac{147}{10}=\frac{8-45+240+147}{10}=\frac{350}{10}=35$$

$$\therefore f(3)=35$$

② Find the third degree polynomial $f(x)$ satisfying the following data:

x :	1	3	5	7
y :	24	120	336	720

Sol: Given $x_0=1$, $x_1=3$, $x_2=5$, $x_3=7$, $y_0=24$, $y_1=120$, $y_2=336$, $y_3=720$.

Lagrange's interpolation formula is

$$y=f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$=\frac{(x-3)(x-5)(x-7)}{(-2)(-4)(-6)}(24)+\frac{(x-1)(x-5)(x-7)}{(2)(-2)(-4)}(120)+\frac{(x-1)(x-3)(x-7)}{(4)(2)(-2)}(336)$$

$$+\frac{(x-1)(x-3)(x-5)}{(6)(4)(2)}(720)$$

$$=\frac{-1}{2}\left[x^3-15x^2+71x-105\right]+\frac{15}{2}\left[x^3-13x^2+47x-35\right]-21\left[x^3-11x^2+31x-21\right]$$

$$+15\left[x^3-9x^2+23x-15\right]$$

$$=x^3+6x^2+11x+6$$

$$\therefore f(x)=x^3+6x^2+11x+6$$

---

③ Find the missing term in the following table using Lagrange's interpolation.

x :	0	1	2	3	4
y :	1	3	9	–	81

Sol: Given $x_0=0$, $x_1=1$, $x_2=2$, $x_3=4$, $y_0=1$, $y_1=3$, $y_2=9$, $y_3=81$

Lagrange's interpolation formula is

$$y=f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$=\frac{(x-1)(x-2)(x-4)}{(-1)(-2)(-4)}(1)+\frac{x(x-2)(x-4)}{(1)(-1)(-3)}(3)+\frac{x(x-1)(x-4)}{(2)(1)(-2)}(9)$$

$$+\frac{x(x-1)(x-2)}{(4)(3)(2)}(81)$$

$$\therefore f(3) = \frac{(2)(1)(-1)}{-8} + \frac{3(1)(-1)}{3}(3) + \frac{(3)(2)(-1)}{-4}(9) + \frac{(3)(2)(1)}{(4)(3)(2)}(81)$$

$$= \frac{1}{4} - 3 + \frac{27}{2} + \frac{81}{4} = \frac{1-12+54+81}{4} = 31$$

$$\therefore f(3) = 31$$

④ Find the parabola of the form $y = ax^2 + bx + c$ passing through the pts $(0,0), (1,1)$ & $(2,20)$.

Sol: Given $x_0 = 0, x_1 = 1, x_2 = 2, y_0 = 0, y_1 = 1, y_2 = 20$.

Lagrange's interpolation formula is

$$y = f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2$$

$$= \frac{(x-1)(x-2)}{(-1)(-2)}(0) + \frac{x(x-2)}{(1)(-1)}(1) + \frac{x(x-1)}{(2)(1)}(20)$$

$$= -x(x-2) + 10x(x-1) = -x^2 + 2x + 10x^2 - 10x = 9x^2 - 8x$$

$$\therefore f(x) = y = 9x^2 - 8x$$

Inverse Interpolation:

The process of finding a value of x for the corresponding value of y is called inverse interpolation. In this case, we will take y as independent variable & x as dependent variable.

Inverse Lagrange's interpolation formula is

$$x = \frac{(y-y_1)(y-y_2)\cdots(y-y_n)}{(y_0-y_1)(y_0-y_2)\cdots(y_0-y_n)} x_0 + \frac{(y-y_0)(y-y_2)\cdots(y-y_n)}{(y_1-y_0)(y_1-y_2)\cdots(y_1-y_n)} x_1$$

$$+ \cdots + \frac{(y-y_0)(y-y_1)\cdots(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)\cdots(y_n-y_{n-1})} x_n$$

Problems:

① Find the age corresponding to the annuity value 13.6 given the table:

Age (x)	: 30	35	40	45	50
Annuity value (y)	: 15.9	14.9	14.1	13.3	12.5

Sol: Given $x_0 = 30, x_1 = 35, x_2 = 40, x_3 = 45, x_4 = 50, y_0 = 15.9, y_1 = 14.9,$

$y_2 = 14.1$, $y_3 = 13.3$, $y_4 = 12.5$

Inverse Lagrange's interpolation formula is

$$x = \frac{(y-y_1)(y-y_2)(y-y_3)(y-y_4)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)(y_0-y_4)} x_0 + \frac{(y-y_0)(y-y_2)(y-y_3)(y-y_4)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)(y_1-y_4)} x_1$$

$$+ \frac{(y-y_0)(y-y_1)(y-y_3)(y-y_4)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)(y_2-y_4)} x_2 + \frac{(y-y_0)(y-y_1)(y-y_2)(y-y_4)}{(y_3-y_0)(y_3-y_1)(y_3-y_2)(y_3-y_4)} x_3$$

$$+ \frac{(y-y_0)(y-y_1)(y-y_2)(y-y_3)}{(y_4-y_0)(y_4-y_1)(y_4-y_2)(y_4-y_3)} x_4$$

$$\therefore x(13.6) = \frac{(-1.3)(-0.5)(0.3)(1.1)}{(1.8)(2.6)(3.4)}(30) + \frac{(-2.3)(-0.5)(0.3)(1.1)}{-(0.8)(1.6)(2.4)}(35)$$

$$+\frac{(-2.3)(-1.3)(0.3)(1.1)}{(1.8)(0.8)(0.8)(1.6)}(40) + \frac{(-2.3)(-1.3)(-0.5)(1.1)}{(-2.6)(1.6)(0.8)(0.8)}(45)$$

$$+\frac{(-2.3)(-1.3)(-0.5)(0.3)}{(3.4)(2.4)(1.6)(0.8)}(50)$$

$$= \frac{55}{136} - \frac{8855}{2048} + \frac{16445}{768} + \frac{56925}{2048} - \frac{22.425}{10.4448} = 43.14$$

$\therefore x(13.6) = 43.14$

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② Find the value of θ given $f(\theta) = 0.3887$ where $f(\theta) = \int_0^\theta \frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}}$ using the table :

θ :	21°	23°	25°
$f(\theta)$:	0.3706	0.4068	0.4433

Sol: Take $\theta = x$ & $f(\theta) = y$

Given $x_0 = 21$, $x_1 = 23$, $x_2 = 25$, $y_0 = 0.3706$, $y_1 = 0.4068$, $y_2 = 0.4433$

Inverse Lagrange's interpolation formula is

$$x = \frac{(y-y_1)(y-y_2)}{(y_0-y_1)(y_0-y_2)} x_0 + \frac{(y-y_0)(y-y_2)}{(y_1-y_0)(y_1-y_2)} x_1 + \frac{(y-y_0)(y-y_1)}{(y_2-y_0)(y_2-y_1)} x_2$$

$$= \frac{(y-0.4068)(y-0.4433)}{(-0.0362)(-0.0727)}(21) + \frac{(y-0.3706)(y-0.4433)}{(0.0362)(-0.0365)}(23) + \frac{(y-0.3706)(y-0.4068)}{(0.0727)(0.0365)}(25)$$

$$\therefore x(0.3887) = \frac{(-0.0181)(-0.0546)}{(-0.0362)(-0.0727)}(21) + \frac{(0.0181)(-0.0546)}{(0.0362)(-0.0365)}(23) + \frac{(0.0181)(-0.0181)}{(0.0727)(0.0365)}(25)$$

$$= 7.8858 + 17.2027 - 3.0865 = 22.002$$

$\therefore \theta(0.3887) = 22.002$

Divided differences:

Let the fun/. $y=f(x)$ take the values $f(x_0), f(x_1), \dots, f(x_n)$ corresponding to the values $x_0, x_1, \dots, x_n$ of the argument x where $x_1-x_0, x_2-x_1, \dots, x_n-x_{n-1}$ need not necessarily be equal.

The first divided difference of $f(x)$ for the arguments x_0, x_1 is

$$f(x_0,x_1) = \frac{f(x_1)-f(x_0)}{x_1-x_0} \cdot \text{|||ly}, \; f(x_1,x_2) = \frac{f(x_2)-f(x_1)}{x_2-x_1} \text{ \& so on.}$$

The second divided difference of $f(x)$ for three arguments x_0, x_1, x_2 is defined as $f(x_0,x_1,x_2) = \dfrac{f(x_1,x_2)-f(x_0,x_1)}{x_2-x_0}$,

$$f(x_1,x_2,x_3) = \frac{f(x_2,x_3)-f(x_1,x_2)}{x_3-x_1} \text{ \& so on.}$$

Properties of Divided differences:

① The divided differences are symmetrical in all their arguments, that is the value of any difference is independent of the order of the arguments.

② The divided difference (of any order) of the sum or difference of two funs/. is equal to the sum or difference of the corresponding separate divided differences.

③ The divided difference of the product of a constant & a fun/. is equal to the product of the constant & the divided difference of the fun/..

④ The n^{th} divided differences of a poly/. of the n^{th} degree are constant.

Problems:

① Form the divided difference table for the following data:

x : 1 2 4 7 12

$f(x)$: 22 30 82 106 206

Sol:

Divided difference table:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1	22	$\frac{30-22}{2-1}=8$	$\frac{26-8}{4-1}=6$	$\frac{-3.6-6}{7-1}=-1.6$	$\frac{0.51+1.6}{12-1}=0.1918$
2	30	$\frac{82-30}{4-2}=26$	$\frac{8-26}{7-2}=-3.6$	$\frac{1.5+3.6}{12-2}=0.51$	
4	82	$\frac{106-82}{7-4}=8$	$\frac{20-8}{12-4}=1.5$		
7	106	$\frac{206-106}{12-7}=20$			
12	206				

② Show that $\Delta^{3}_{bcd}\left(\frac{1}{a}\right) = \frac{-1}{abcd}$

Sol: Given $f(a) = \frac{1}{a}$

$$f(a,b) = \frac{f(b)-f(a)}{b-a} = \frac{\frac{1}{b}-\frac{1}{a}}{b-a} = \frac{a-b}{ab(b-a)} = \frac{-1}{ab}$$

$$f(a,b,c) = \frac{f(b,c)-f(a,b)}{c-a} = \frac{\frac{-1}{bc}+\frac{1}{ab}}{c-a} = \frac{-a+c}{abc(c-a)} = \frac{1}{abc}$$

$$f(a,b,c,d) = \frac{f(b,c,d)-f(a,b,c)}{d-a} = \frac{\frac{1}{bcd}-\frac{1}{abc}}{d-a} = \frac{a-d}{abcd(d-a)} = \frac{-1}{abcd}$$

$$\therefore f(a,b,c,d) = \Delta^{3}_{bcd}\left(\frac{1}{a}\right) = \frac{-1}{abcd}$$

Newton's divided difference interpolation formula for unequal intervals:

$$f(x) = f(x_0) + (x-x_0)f(x_0,x_1) + (x-x_0)(x-x_1)f(x_0,x_1,x_2) + \dots + (x-x_0)(x-x_1)\dots(x-x_{n-1})f(x_0,x_1,\dots,x_n).$$

Problems:

① Find $f(x)$ as a poly. in x for the following data by Newton's divided difference formula:

x:	-4	-1	0	2	5
$f(x)$:	1245	33	5	9	1335

Sol: Divided difference table:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
-4	1245				
		$\frac{33-1245}{-1+4} = -404$			
-1	33		$\frac{-28+404}{0+4} = 94$		
		$\frac{5-33}{0+1} = -28$		$\frac{10-94}{2+4} = -14$	
0	5		$\frac{2+28}{2+1} = 10$		$\frac{13+14}{5+4} = 3$
		$\frac{9-5}{2-0} = 2$		$\frac{88-10}{5+1} = 13$	
2	9		$\frac{442-2}{5-0} = 88$		
		$\frac{1335-9}{5-2} = 442$			
5	1335				

Newton's divided difference interpolation formula is

$$f(x) = f(x_0) + (x-x_0)f(x_0,x_1) + (x-x_0)(x-x_1)f(x_0,x_1,x_2) + (x-x_0)(x-x_1)(x-x_2)f(x_0,x_1,x_2,x_3) + (x-x_0)(x-x_1)(x-x_2)(x-x_3)f(x_0,x_1,x_2,x_3,x_4)$$

Here $x_0 = -4$, $x_1 = -1$, $x_2 = 0$, $x_3 = 2$, $x_4 = 5$

$f(x_0) = 1245$, $f(x_0, x_1) = -404$, $f(x_0, x_1, x_2) = 94$, $f(x_0, x_1, x_2, x_3) = -14$,

$f(x_0, x_1, x_2, x_3, x_4) = 3$.

$$\therefore f(x) = 1245 + (x+4)(-404) + (x+4)(x+1)(94) + (x+4)(x+1)x(-14) + (x+4)(x+1)x(x-2)(3)$$

$$= 1245 - 404x - 1616 + 94x^2 + 470x + 376 - 14x^3 - 70x^2 - 56x + 3x^4 + 15x^3 + 12x^2 - 6x^3 - 30x^2 - 24x$$

$$\therefore f(x) = 3x^4 - 5x^3 + 6x^2 - 14x + 5$$

---

② Using Newton's divided difference formula find the missing value from the table:

x:	1	2	4	5	6
y:	14	15	5	-	9

Sol: Divided difference table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1	14			
		$\frac{15-14}{2-1} = 1$		
2	15		$\frac{-5-1}{4-1} = -2$	
		$\frac{5-15}{4-2} = -5$		$\frac{\frac{7}{4}+2}{6-1} = \frac{3}{4}$
4	5		$\frac{2+5}{6-2} = \frac{7}{4}$	
		$\frac{9-5}{6-4} = 2$		
6	9			

Here $x_0 = 1$, $x_1 = 2$, $x_2 = 4$, $x_3 = 6$

$f(x_0) = 14$, $f(x_0, x_1) = 1$

$f(x_0, x_1, x_2) = -2$

$f(x_0, x_1, x_2, x_3) = \frac{3}{4}$

Newton's divided difference formula is

$$y = f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) + (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3)$$

$$= 14 + (x-1)(1) + (x-1)(x-2)(-2) + (x-1)(x-2)(x-4)\left(\frac{3}{4}\right)$$

$$\therefore f(5) = 14 + 4 + (4)(3)(-2) + (4)(3)(1)\left(\frac{3}{4}\right) = 3$$

Newton's forward interpolation formula for equal intervals: (Gregory-Newton forward interpolation formula)

$$y(x) = y_0 + \frac{u}{1!} \Delta y_0 + \frac{u(u-1)}{2!} \Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!} \Delta^3 y_0 + \dots + \frac{u(u-1)(u-2) \dots (u-(n-1))}{n!} \Delta^n y_0$$

where $u = \frac{x - x_0}{h}$.

Newton's backward interpolation formula for equal intervals: (Gregory-Newton backward interpolation formula)

$$y(x) = y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n$$

$$+ \dots + \frac{v(v+1)(v+2)\dots(v+(n-1))}{n!}\nabla^n y_n \quad \text{where } v = \frac{x-x_n}{h}$$

Problems:

① Using Newton's forward interpolation formula, find the polyl. $f(x)$ satisfying the following data. Hence evaluate y at $x=5$.

x :	4	6	8	10
y :	1	3	8	10

Sol: Difference table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
4	1	$2=3-1$	$5-2=3$	$-3-3=-6$
6	3	$5=8-3$	$2-5=-3$	
8	8	$2=10-8$		
10	10			

Here $x_0=4$, $h=2$

$y_0=1$, $\Delta y_0 = 2$

$\Delta^2 y_0 = 3$, $\Delta^3 y_0 = -6$

$u = \frac{x-x_0}{h} = \frac{x-4}{2}$

Newton's forward interpolation formula is

$$y(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

$$= 1 + \frac{\left(\frac{x-4}{2}\right)}{1!}(2) + \frac{\left(\frac{x-4}{2}\right)\left(\frac{x-6}{2}\right)}{2!}(3) + \frac{\left(\frac{x-4}{2}\right)\left(\frac{x-6}{2}\right)\left(\frac{x-8}{2}\right)}{3!}(-6)$$

$$= 1 + x - 4 + \frac{3}{8}(x-4)(x-6) - \frac{1}{8}(x-4)(x-6)(x-8)$$

$$= -3 + x + \frac{3}{8}(x^2-10x+24) - \frac{1}{8}(x^3-10x^2+24x-8x^2+80x-192)$$

$$= \frac{1}{8}\left[-24+8x+3x^2-30x+72-x^3+18x^2-104x+192\right]$$

$$\therefore y(x) = \frac{1}{8}\left[-x^3+21x^2-126x+240\right]$$

$$\therefore y(5) = \frac{1}{8}\left[-5^3+21(5)^2-126(5)+240\right] = 1.25$$

---

② Using Newton's forward interpolation formula find the cubic polyl. which takes places the following values:

x :	0	1	2	3
$f(x)$:	1	2	1	10

Evaluate $f(4)$ using Newton's backward formula. Is it the same as obtained from the cubic polyl. found above.

Sol: Difference table:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	1			
		$2-1=1$		
1	2		$-1-1=-2$	
		$1-2=-1$		$10+2=12$
2	1		$9+1=10$	
		$10-1=9$		
3	10			

Here $x_0=0$, $h=1$

$y_0=1$, $\Delta y_0=1$, $\Delta^2 y_0=-2$,

$\Delta^3 y_0=12$

$x_3=3$, $y_3=10$, $\nabla y_3=9$,

$\nabla^2 y_3=10$, $\nabla^3 y_3=12$

Newton's forward interpolation formula is

$$f(x)=y_0+\frac{u}{1!}\Delta y_0+\frac{u(u-1)}{2!}\Delta^2 y_0+\frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 \text{ where } u=\frac{x-x_0}{h}=\frac{x-0}{1}=x$$

$$=1+\frac{x}{1!}(1)+\frac{x(x-1)}{2!}(-2)+\frac{x(x-1)(x-2)}{3!}(12)$$

$$=1+x-x^2+x+2(x(x^2-3x+2))=-x^2+2x+1+2x^3-6x^2+4x$$

$$=2x^3-7x^2+6x+1 \quad — ①$$

Newton's backward interpolation formula is

$$f(x)=y_3+\frac{v}{1!}\nabla y_3+\frac{v(v+1)}{2!}\nabla^2 y_3+\frac{v(v+1)(v+2)}{3!}\nabla^3 y_3 \text{ where } v=\frac{x-x_3}{h}=x-3$$

$$=10+(x-3)(9)+\frac{(x-3)(x-2)}{2}(10)+\frac{(x-3)(x-2)(x-1)}{6}(12)$$

$$=10+9x-27+5(x^2-5x+6)+2(x^3-5x^2+6x-x^2+5x-6)$$

$$=2x^3-7x^2+6x+1 \quad — ②$$

From ① & ②, we have Newton's forward & backward difference polys/. are same.

$$\therefore f(4)=2(4)^3-7(4)^2+6(4)+1=41$$

---

③ From the following data, find θ at $x=43$ & $x=84$

x :	40	50	60	70	80	90
θ :	184	204	226	250	276	304

Also express θ interms of x.

Sol: Difference table:

x	θ	$\Delta\theta$	$\Delta^2\theta$	$\Delta^3\theta$
40	184			
		20		
50	204		2	
		22		0
60	226		2	
		24		0
70	250		2	
		26		0
80	276		2	
		28		
90	304			

Here $x_0=40$, $h=10$

$\theta_0=184$, $\Delta\theta_0=20$, $\Delta^2\theta_0=2$

$x_5=90$, $\theta_5=304$, $\nabla\theta_5=28$,

$\nabla^2\theta_5=2$

To find θ at $x=43$, use Newton's forward interpolation & to find θ at $x=84$, use Newton's backward interpolation formula.

Newton's forward interpolation formula is

$$\theta(x) = \theta_0 + \frac{u}{1!}\Delta\theta_0 + \frac{u(u-1)}{2!}\Delta^2\theta_0 \quad \text{where } u = \frac{\cancel{x-x_0}}{h} = \frac{x-40}{10}$$

$$= 184 + \left(\frac{x-40}{10}\right)(20) + \frac{\left(\frac{x-40}{10}\right)\left(\frac{x-50}{10}\right)}{2!}(2)$$

$$= 184 + 2(x-40) + \frac{1}{100}(x-40)(x-50)$$

$$= 184 + 2x - 80 + \frac{x^2}{100} - \frac{90x}{100} + \frac{2000}{100} = 104 + 2x + 0.01x^2 - 0.9x + 20$$

$$\therefore \theta(x) = 0.01x^2 + 1.1x + 124$$

$$\therefore \theta(43) = 0.01(43)^2 + 1.1(43) + 124 = 189.79$$

Newton's backward interpolation formula is

$$\theta(x) = \theta_5 + \frac{v}{1!}\nabla\theta_5 + \frac{v(v+1)}{2!}\nabla^2\theta_5 \quad \text{where } v = \frac{x-x_5}{h} = \frac{84-90}{10} = -0.6$$

$$\theta(84) = 304 + (-0.6)(28) + \frac{(-0.6)(0.4)}{2}(2) = 286.96$$

---

④ From the data given below, find the no. of students whose weight is between 60 to 70.

Weight :	0-40	40-60	60-80	80-100	100-120
No. of students :	250	120	100	70	50

Sol. Difference table:

x (Weight)	y (No. of Students)	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
Below 40	250				
		120			
Below 60	370		−20		
		100		−10	
Below 80	470		−30		20
		70		10	
Below 100	540		−20		
		50			
Below 120	590				

Here $x_0 = 40, h = 20$

$y_0 = 250$, $\Delta y_0 = 120$,

$\Delta^2 y_0 = -20$,

$\Delta^3 y_0 = -10$, $\Delta^4 y_0 = 20$

Let us calculate the no. of students whose weight is less than 70.

Newton's forward interpolation formula is

$$y(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0$$

$$\text{where } u = \frac{x-x_0}{h} = \frac{70-40}{20} = 1.5$$

$$= 250 + (1.5)(120) + \frac{(1.5)(0.5)}{2}(-20) + \frac{(1.5)(0.5)(-0.5)}{6}(-10) + \frac{(1.5)(0.5)(-0.5)(-1.5)}{24}(20)$$

$$= 250 + 180 - 7.5 + 0.625 + 0.46875 = 423.5938 = 424$$

No. of students b/w 60 to 70 $= 424 - 370 = \underline{\underline{54}}$

Difference operators & relations:

Forward difference operator (Δ): $\Delta f(x) = f(x+h) - f(x)$

Backward difference operator (∇): $\nabla f(x) = f(x) - f(x-h)$

Central difference operator (δ): $\delta f(x) = f(x+\frac{h}{2}) - f(x-\frac{h}{2})$

Shifting or displacement or translation operator (E): $Ef(x) = f(x+h)$

Relation between Δ & E:

$$\Delta f(x) = f(x+h) - f(x)$$
$$= Ef(x) - f(x) = (E-1)f(x)$$
$$\therefore \boxed{\Delta = E - 1} \Rightarrow \boxed{E = \Delta + 1}$$

Relation between ∇ & E:

$$\boxed{\begin{array}{l} E f(x-h) = f(x-h+h) \\ E f(x-h) = f(x) \\ \Rightarrow f(x-h) = E^{-1} f(x) \end{array}}$$

$$\nabla f(x) = f(x) - f(x-h)$$
$$= f(x) - E^{-1}f(x) = (1-E^{-1})f(x)$$
$$\therefore \boxed{\nabla = 1 - E^{-1}} \Rightarrow E^{-1} = 1 - \nabla \Rightarrow \boxed{E = (1-\nabla)^{-1}}$$

Relation between E & δ:

$$\delta f(x) = f(x+\frac{h}{2}) - f(x-\frac{h}{2})$$
$$= E^{1/2} f(x) - E^{-1/2} f(x)$$
$$= (E^{1/2} - E^{-1/2}) f(x)$$
$$\therefore \delta = E^{1/2} - E^{-1/2} = E^{1/2} - \frac{1}{E^{1/2}} = E^{1/2}\left[1 - \frac{1}{E}\right] = E^{1/2}\left[1 - E^{-1}\right]$$
$$\delta = E^{1/2}\nabla$$
$$\delta = \frac{1}{E^{1/2}}\left[E - 1\right] = E^{-1/2}\Delta$$
$$\therefore \boxed{\delta = E^{1/2} - E^{-1/2} = E^{1/2}\nabla = E^{-1/2}\Delta}$$

NUMERICAL DIFFERENTIATION AND INTEGRATION

Derivatives using Newton's forward interpolation:

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2}\Delta^2 y_0 + \frac{3u^2-6u+2}{6}\Delta^3 y_0 + \frac{4u^3-18u^2+22u-6}{24}\Delta^4 y_0 + \cdots\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + (u-1)\Delta^3 y_0 + \frac{6u^2-18u+11}{12}\Delta^4 y_0 + \cdots\right]$$

$$\frac{d^3y}{dx^3} = \frac{1}{h^3}\left[\Delta^3 y_0 + \frac{12u-18}{12}\Delta^4 y_0 + \cdots\right]$$

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right] \text{—①}$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \cdots\right] \text{—②}$$

$$\left(\frac{d^3y}{dx^3}\right)_{x=x_0} = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \cdots\right] \text{—③}$$

Eqns). ①, ② & ③ give the values of first, second & third derivatives at the starting value $x=x_0$.

Derivatives using Newton's backward interpolation:

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{2v+1}{2}\nabla^2 y_n + \frac{3v^2+6v+2}{6}\nabla^3 y_n + \frac{4v^3+18v^2+22v+6}{24}\nabla^4 y_n + \cdots\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + (v+1)\nabla^3 y_n + \frac{6v^2+18v+11}{12}\nabla^4 y_n + \cdots\right]$$

$$\frac{d^3y}{dx^3} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{12v+18}{12}\nabla^4 y_n + \cdots\right]$$

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \cdots\right] \text{—①}$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \cdots\right] \text{—②}$$

$$\left(\frac{d^3y}{dx^3}\right)_{x=x_n} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \cdots\right] \text{—③}$$

Eqns). ①, ② & ③ give the values of first, second & third derivatives at the ending value $x=x_n$.

Derivatives using Stirling's formula:

Stirling's formula is

$$y(x) = y_0 + \frac{u}{2}\left[\Delta y_0 + \Delta y_{-1}\right] + \frac{u^2}{2}\Delta^2 y_{-1} + \frac{u^3-u}{12}\left[\Delta^3 y_{-1} + \Delta^3 y_{-2}\right] + \frac{u^4-u^2}{24}\Delta^4 y_{-2} + \cdots$$

where $u = \frac{x-x_0}{h}$

$$\frac{dy}{dx}=\frac{1}{h}\left[\frac{1}{2}(\Delta y_0+\Delta y_{-1})+u\Delta^2 y_{-1}+\frac{3u^2-1}{12}(\Delta^3 y_{-1}+\Delta^3 y_{-2})+\frac{4u^3-2u}{24}\Delta^4 y_{-2}+\cdots\right]$$

$$\frac{d^2y}{dx^2}=\frac{1}{h^2}\left[\Delta^2 y_{-1}+\frac{u}{2}(\Delta^3 y_{-1}+\Delta^3 y_{-2})+\frac{6u^2-1}{12}\Delta^4 y_{-2}+\cdots\right]$$

$$\frac{d^3y}{dx^3}=\frac{1}{h^3}\left[\frac{1}{2}(\Delta^3 y_{-1}+\Delta^3 y_{-2})+u\Delta^4 y_{-2}+\cdots\right]$$

$$\left(\frac{dy}{dx}\right)_{x=x_0}=\frac{1}{h}\left[\frac{1}{2}(\Delta y_0+\Delta y_{-1})-\frac{1}{12}(\Delta^3 y_{-1}+\Delta^3 y_{-2})+\cdots\right] \text{ — ①}$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0}=\frac{1}{h^2}\left[\Delta^2 y_{-1}-\frac{1}{12}\Delta^4 y_{-2}+\cdots\right] \text{ — ②}$$

$$\left(\frac{d^3y}{dx^3}\right)_{x=x_0}=\frac{1}{h^3}\left[\frac{1}{2}(\Delta^3 y_{-1}+\Delta^3 y_{-2})+\cdots\right] \text{ — ③}$$

Eqns). ①, ② & ③ give the values of first, second & third derivatives at the starting value $x=x_0$.

Derivatives using Bessel's formula:

$$y(x)=\frac{1}{2}(y_0+y_1)+\left(u-\frac{1}{2}\right)\Delta y_0+\frac{u(u-1)}{4}(\Delta^2 y_{-1}+\Delta^2 y_0)+\frac{u(u-\frac{1}{2})(u-1)}{6}\Delta^3 y_{-1}$$
$$+\frac{(u+1)u(u-1)(u-2)}{48}(\Delta^4 y_{-2}+\Delta^4 y_{-1})+\cdots$$

$$y'(x)=\frac{1}{h}\left[\Delta y_0+\frac{2u-1}{4}(\Delta^2 y_{-1}+\Delta^2 y_0)+\frac{3u^2-3u+\frac{1}{2}}{6}\Delta^3 y_{-1}+\cdots\right]$$

Problems:

① Find $f'(3)$ & $f''(3)$ for the following data:

x:	3	3.2	3.4	3.6	3.8	4
$f(x)$:	-14	-10.032	-5.296	-0.256	6.672	14

Sol: Difference table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
3	-14					
		3.968				
			0.768			
3.2	-10.032			-0.464		
		4.736			2.048	
3.4	-5.296		0.304			-5.12
		5.04		1.584		
3.6	-0.256		1.888		-3.072	
		6.928		-1.488		
3.8	6.672		0.4			
		7.328				
4	14					

By Newton's forward differentiation formula,

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right]$$

$$= 9.4667$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \cdots\right]$$

$$= 184.4$$

---

② Find $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$ at $x=51$ from the following data:

x:	50	60	70	80	90
y:	19.96	36.65	58.81	77.21	94.61

Sol: Here $h=10$. To find the derivatives of y at $x=51$ we use Newton's forward differentiation formula taking the origin at $x_0=50$.

We have $u = \frac{x-x_0}{h} = \frac{51-50}{10} = \frac{1}{10} = 0.1$

Difference table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
50	19.96				
		16.69			
60	36.65		5.47		
		22.16		-9.23	
70	58.81		-3.76		11.99
		18.4		2.76	
80	77.21		-1		
		17.4			
90	94.61				

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2}\Delta^2 y_0 + \frac{3u^2-6u+2}{6}\Delta^3 y_0 + \frac{4u^3-18u^2+22u-6}{24}\Delta^4 y_0 + \cdots\right]$$

$$= 1.0316$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + (u-1)\Delta^3 y_0 + \frac{6u^2-18u+11}{12}\Delta^4 y_0 + \cdots\right] = 0.2303$$

---

③ Find the maximum & minimum value of y tabulated below.

x:	-2	-1	0	1	2	3	4
y:	2	-0.25	0	-0.25	2	15.75	56

Sol: Difference table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
-2	2					
		-2.25				
-1	-0.25		2.5			
		0.25		-3		
0	0		-0.5		6	
		$\boxed{-0.25}$		3		0
1	-0.25		$\boxed{2.5}$		6	
		2.25		$\boxed{9}$		0
2	2		11.5		$\boxed{6}$	
		13.75		15		
3	15.75		26.5			
		40.25				
4	56					

Here $x_0 = 0$

$u = \frac{x-x_0}{h} = x$

Newton's forward formula for derivatives is

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2}\Delta^2 y_0 + \frac{3u^2-6u+2}{6}\Delta^3 y_0 + \frac{4u^3-18u^2+22u-6}{24}\Delta^4 y_0 + \cdots\right]$$

$$= -0.25 + \frac{(2x-1)}{2}(2.5) + \frac{3x^2-6x+2}{6}(9) + \frac{4x^3-18x^2+22x-6}{24}(6)$$

$$= -0.25 + 2.5x - 1.25 + 4.5x^2 - 9x + 3 + x^3 - 4.5x^2 + 5.5x - 1.5$$

$$= x^3 - x$$

Now $\frac{dy}{dx} = 0 \Rightarrow x^3 - x = 0 \Rightarrow x(x^2-1) = 0 \Rightarrow x = 0, 1, -1$

$$\frac{d^2y}{dx^2} = 3x^2 - 1$$

At $x=0$, $\frac{d^2y}{dx^2} = -1 \rightarrow -ve$; At $x=1$, $\frac{d^2y}{dx^2} = 2 \rightarrow +ve$

At $x=-1$, $\frac{d^2y}{dx^2} = 2 \rightarrow +ve$

$\therefore$ y is maximum at $x=0$, minimum at $x = 1$ & -1.

$$\therefore y(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

$$= 0 + x(-0.25) + \frac{x(x-1)}{2}(2.5) + \frac{x(x-1)(x-2)}{6}(9) + \frac{x(x-1)(x-2)(x-3)}{24}(6) + \cdots$$

$y(0) = 0$, $y(1) = -0.25$

$\therefore$ Maximum value of $y(x)$ is 0 & minimum value of $y(x)$ is -0.25

④ Consider the following table of data:

x:	0.2	0.4	0.6	0.8	1.0
$f(x)$:	0.9798652	0.9177710	0.8080348	0.6386093	0.3843735

Find $f'(0.25)$ using Newton's forward difference approximation & $f'(0.95)$ using Newton's backward difference approximation.

Sol: Difference table:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0.2	0.9798652				
		-0.0620942			
0.4	0.9177710		-0.047642		
		-0.1097362		-0.0120473	
0.6	0.8080348		-0.0596893		-0.0130737
		-0.1694255		-0.025121	
0.8	0.6386093		-0.0848103		
		-0.2542358			
1	0.3843735				

Here $h = 0.2$

Newton's forward interpolation formula for derivative

$$y'(x) = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2}\Delta^2 y_0 + \frac{3u^2-6u+2}{6}\Delta^3 y_0 + \frac{4u^3-18u^2+22u-6}{24}\Delta^4 y_0 + \cdots\right]$$

where $u = \frac{x-x_0}{h} = \frac{0.25-0.2}{0.2} = 0.25$

$$\therefore y'(0.25) = \frac{1}{0.2}\left[-0.0620942 + 0.0119105 - 0.0013804198 + 0.00085115523\right]$$

$= -0.2536$

Newton's backward difference formula for derivative

$$y'(x) = \frac{1}{h}\left[\nabla y_n + \frac{2v+1}{2}\nabla^2 y_n + \frac{3v^2+6v+2}{6}\nabla^3 y_n + \frac{4v^3+18v^2+22v+6}{24}\nabla^4 y_n + \cdots\right]$$

where $v = \frac{x-x_n}{h} = \frac{0.95-1}{0.2} = -0.25$

$$= \frac{1}{0.2}\left[-0.2542358 - 0.0212026 - 0.0028784 - 0.0008512\right]$$

$= -1.3958$

⑤ Consider the following table of data:

x :	0.2	0.4	0.6	0.8	1
$f(x)$:	0.9798652	0.9177710	0.8080348	0.6386093	0.3843735

Find $f'(0.6)$ using Stirling's approximation.

Sol: Difference table:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0.2	0.9798652				
		−0.0620942			
0.4	0.9177710		−0047642		
		$\boxed{-0.1097362}$		$\boxed{-0.0120473}$	
0.6	$\boxed{0.8080348}$		−0.0596893		−0.0130737
		$\boxed{-0.1694255}$		$\boxed{-0.025121}$	
0.8	0.6386093		−0.0848103		
		−0.2542358			
1	0.3843735				

Stirling's formula for derivative

$$y'(x) = \frac{1}{h}\left[\frac{1}{2}(\Delta y_0 + \Delta y_{-1}) - \frac{1}{12}(\Delta^3 y_{-1} + \Delta^3 y_{-2}) + \cdots\right]$$ where $u = \frac{x-x_0}{h} = \frac{0.6-0.6}{0.2} = 0$

$= -0.6824$

⑥ Obtain the value of $f'(0.04)$ using Bessel's formula given in the table below:

x :	0.01	0.02	0.03	0.04	0.05	0.06
$f(x)$:	0.1023	0.1047	0.1071	0.1096	0.1122	0.1148

Sol: Difference table:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
0.01	0.1023					
		0.0024				
0.02	0.1047		0			
		0.0024		0.0001		
0.03	0.1071		0.0001		−0.0001	
		0.0025		0		0
0.04	0.1096		$\boxed{0.0001}$		$\boxed{-0.0001}$	
		$\boxed{0.0026}$		$\boxed{-0.0001}$		
0.05	0.1122		$\boxed{0}$			
		0.0026				
0.06	0.1148					

Bessel's formula for derivative is

$$y'(x) = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{4}(\Delta^2 y_{-1} + \Delta^2 y_0) + \frac{3u^2-3u+\frac{1}{2}}{6}\Delta^3 y_{-1} + \frac{4u^3-6u^2+2u+2}{48}(\Delta^4 y_{-2} + \Delta^4 y_{-1}) + \cdots\right]$$

where $u = \frac{x-x_0}{h} = \frac{0.04-0.04}{0.01} = 0$

$\therefore y'(0.04) = 0.2563$

⑦ Given the following data, find $y'(6)$ & the maximum value of y.

x :	0	2	3	4	7	9
y :	4	26	58	112	466	922

Sol: Since the values of x are not equally spaced, we will use Newton's divided difference formula.

Divided difference table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	4				
		11			
2	26		7		
		32		1	
3	58		11		0
		54		1	
4	112		16		0
		118		1	
7	466		22		
		228			
9	922				

By Newton's divided difference formula,

$$y = f(x) = f(x_0) + (x-x_0)f(x_0,x_1) + (x-x_0)(x-x_1)f(x_0,x_1,x_2) + (x-x_0)(x-x_1)(x-x_2)f(x_0,x_1,x_2,x_3) + \cdots$$

$$= 4 + x(11) + x(x-2)(7) + x(x-2)(x-3)(1)$$

$$= x^3 + 2x^2 + 3x + 4$$

$\therefore y'(x) = 3x^2 + 4x + 3$

$y'(6) = 3(36)+24+3 = 135$

Now $y'(x)=0 \Rightarrow 3x^2+4x+3=0 \Rightarrow x = \dfrac{-4 \pm \sqrt{16-4(3)(3)}}{2(3)}$

$$\Rightarrow x = \frac{-4 \pm 2i\sqrt{5}}{6} = \frac{-2 \pm i\sqrt{5}}{3}$$

Here the roots are imaginary. ∴ There is no extreme value in the range.

⑧ From the following table of values of x & y find $\frac{dy}{dx}$ at $x = 900$.

x:	0	300	600	900	1200	1500	1800
y:	135	149	157	183	201	205	193

Sol: Difference table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
0	135						
		14					
300	149		-6				
		8		24			
600	157		18		-50		
		$\boxed{26}$		$\boxed{-26}$		$\boxed{70}$	
$\boxed{900}$	183		-8		20		-86
		$\boxed{18}$		$\boxed{-6}$		$\boxed{-16}$	
1200	201		-14		4		
		4		-2			
1500	205		-16				
		-12					
1800	193						

Since $x = 900$ is in the middle of the table we use Stirling's formula.

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\frac{1}{2}(\Delta y_0 + \Delta y_{-1}) - \frac{1}{12}(\Delta^3 y_{-1} + \Delta^3 y_{-2}) + \frac{1}{60}(\Delta^5 y_{-2} + \Delta^5 y_{-3}) + \cdots\right]$$

$$= \frac{1}{300}\left[\frac{1}{2}(18+26) - \frac{1}{12}(-6-26) + \frac{1}{60}(-16+70)\right]$$

$= 0.0852$

Numerical Integration:

The process of evaluating a definite integral from a set of tabulated values of the integrand $f(x)$ is called numerical integration. This process when applied to a funˢ. of a single variable, is known as quadratic.

Newton - Cotes quadrature formula:

$$\int_{x_0}^{x_0+nh} f(x)dx = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0\right.$$
$$+\left(\frac{n^4}{5} - \frac{3n^3}{2} + \frac{11n^2}{3} - 3n\right)\frac{\Delta^4 y_0}{4!} + \left(\frac{n^5}{6} - 2n^4 + \frac{35n^3}{4} - \frac{50n^2}{3} + 12n\right)\frac{\Delta^5 y_0}{5!}$$
$$\left. + \left(\frac{n^6}{7} - \frac{15n^5}{6} + 17n^4 - \frac{225n^3}{4} + \frac{274n^2}{3} - 60n\right)\frac{\Delta^6 y_0}{6!} + \cdots\right]$$

Trapezoidal rule:

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{2}\left[\text{Sum of the first \& last ordinates} + 2(\text{Sum of the remaining ordinates})\right]$$
$$= \frac{h}{2}\left[(y_0+y_n) + 2(y_1+y_2+\cdots+y_{n-1})\right]$$

Simpson's 1/3rd rule:

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{3}\left[\text{Sum of the first \& last ordinates} + 2(\text{Sum of remaining odd ordinates}) + 4(\text{Sum of even ordinates})\right]$$
$$= \frac{h}{3}\left[(y_0+y_n) + 4(y_1+y_3+\cdots+y_{n-1}) + 2(y_2+y_4+\cdots+y_{n-2})\right]$$

Note: While applying the above eqn., the given interval must be divided into even no. of equal sub-intervals.

Simpson's 3/8th rule:

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{3h}{8}\left[(y_0+y_n) + 3(y_1+y_2+y_4+y_5+\cdots+y_{n-1}) + 2(y_3+y_6+\cdots+y_{n-3})\right]$$

Note: While applying the above eqn., the no. of sub-intervals should be taken multiple of 3.

Problems:

① Using Trapezoidal rule, evaluate $\int_{-1}^{1} \frac{dx}{1+x^2}$ taking 8 intervals.

Sol: Here $y(x) = \frac{1}{1+x^2}$

Length of the interval = 2

So we divide 8 equal intervals with $h=\frac{2}{8}=0.25$

x :	-1	-0.75	-0.5	-0.25	0	0.25	0.5	0.75	1
y :	0.5	0.64	0.8	0.9412	1	0.9412	0.8	0.64	0.5

Trapezoidal rule is

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left[(y_0+y_8)+2(y_1+y_2+\cdots+y_7)\right]$$

$$\therefore \int_{-1}^{1}\frac{dx}{1+x^2} = \frac{0.25}{2}(12.5248)=\underline{1.5656}$$

② Using Simpson's $\frac{1}{3}^{rd}$ rule evaluate $\int_0^1 xe^x\,dx$ taking 4 intervals. Compare your result with actual value.

Sol: Here $y(x)=xe^x$, $h=\frac{1}{4}=0.25$

x :	0	0.25	0.5	0.75	1
y :	0	0.321	0.8244	1.5878	2.7183

Simpson's $\frac{1}{3}^{rd}$ rule is

$$\int_0^1 xe^x\,dx = \frac{h}{3}\left[(y_0+y_4)+4(y_1+y_3)+2(y_2)\right] = 1.0002$$

Actual Value:

$$\int_0^1 xe^x\,dx = \left[xe^x-e^x\right]_0^1 = (e^1-e^1)-(0-e^0) = \underline{1}$$

③ Calculate $\int_{0.5}^{0.7} e^{-x}\sqrt{x}\,dx$ taking 5 ordinates by Simpson's $\frac{1}{3}^{rd}$ rule.

Sol: Given $y(x)=e^{-x}\sqrt{x}$, $h=\frac{0.2}{4}=0.05$

x :	0.5	0.55	0.6	0.65	0.7
y :	0.4289	0.4279	0.4251	0.4209	0.4155

By Simpson's $\frac{1}{3}^{rd}$ rule,

$$\int_{0.5}^{0.7} e^{-x}\sqrt{x}\,dx = \frac{h}{3}\left[(y_0+y_4)+4(y_1+y_3)+2(y_2)\right] = 0.0848$$

④ Find the value of $\log 2^{1/3}$ from $\int_0^1 \frac{x^2}{1+x^3}\,dx$ using Simpson's $\frac{1}{3}^{rd}$ rule with $h=0.25$.

Sol: Given $y(x)=\frac{x^2}{1+x^3}$, $h=0.25$

x :	0	0.25	0.5	0.75	1
y :	0	0.0615	0.2222	0.3956	0.5

Simpson's $\frac{1}{3}$rd rule is

$$\int_0^1 \frac{x^2}{1+x^3}\,dx = \frac{h}{3}\left[(y_0+y_4)+4(y_1+y_3)+2(y_2)\right] = 0.2311 \text{ — ①}$$

Actual value:

$$\int_0^1 \frac{x^2}{1+x^3}\,dx = \frac{1}{3}\int_0^1 \frac{3x^2}{1+x^3}\,dx = \frac{1}{3}\left[\log(1+x^3)\right]_0^1 = \frac{1}{3}(\log 2-\log 1)$$

$$= \frac{1}{3}\log 2 = \log 2^{1/3} = 0.2311 \;(\because \text{by ①})$$

⑤ Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by (i) Trapezoidal rule (ii) Simpson's rule. Also check up the results by actual integration.

Sol: Here $y(x) = \frac{1}{1+x^2}$. Divide into 6 equal parts. $h = \frac{6}{6} = 1$

x :	0	1	2	3	4	5	6
$y(x)$:	1	0.5	0.2	0.1	0.0588	0.0385	0.027

(i) Trapezoidal rule:

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{2}\left[(y_0+y_6)+2(y_1+y_2+\cdots+y_5)\right] = 1.4108$$

(ii) Simpson's $\frac{1}{3}$rd rule:

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{3}\left[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)\right] = 1.3662$$

Simpson's $\frac{3}{8}$th rule:

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3h}{8}\left[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2(y_3)\right] = 1.3571$$

Actual value:

$$\int_0^6 \frac{dx}{1+x^2} = \left[\tan^{-1}x\right]_0^6 = \tan^{-1}6-\tan^{-1}0 = \tan^{-1}6-0 = \tan^{-1}6 = 1.4056$$

Here the value by Trapezoidal rule is closer to the actual value than the value by Simpson's rule.

⑥ The velocity v of a particle at a distance S from a pt on its path is given by the table below.

S in metre :	0	10	20	30	40	50	60
v m/sec :	47	58	64	65	61	52	38

1,13,14,19,32,36,49

Estimate the time taken to travel 60 metres by using Simpson's $\frac{1}{3}$rd rule. Compare your answer with Simpson's $\frac{3}{8}$th rule.

Sol: WKT $v = \frac{ds}{dt} \Rightarrow dt = \frac{ds}{v} \Rightarrow t = \int_0^{60} \frac{ds}{v}$. Take $y = \frac{1}{v}$

s :	0	10	20	30	40	50	60
$y = \frac{1}{v}$:	0.0213	0.0172	0.0156	0.0154	0.0164	0.0192	0.0263

Simpson's $\frac{1}{3}$rd rule is

$$t = \int_0^{60} \frac{ds}{v} = \frac{h}{3}\left[(y_0+y_6)+2(y_2+y_4)+4(y_1+y_3+y_5)\right]$$

$$= \frac{10}{3}\left[0.0476+0.064+0.2072\right] = 1.0627 \text{ sec}$$

Simpson's $\frac{3}{8}$th rule is

$$t = \int_0^{60} \frac{ds}{v} = \frac{3h}{8}\left[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2(y_3)\right] = 1.0635 \text{ sec}$$

⑦ The velocity v starts from rest is given at fixed intervals of time t as follows:

t :	2	4	6	8	10	12	14	16	18	20
v :	10	18	25	29	32	20	11	5	2	0

Estimate the distance covered in 20 mins by Simpson's rule.

Sol: Since it starts from rest at $t=0$, $v=0$. Introducing this value we get 11 ordinates & use Simpson's $\frac{1}{3}$rd rule.

$$v = \frac{ds}{dt} \Rightarrow v\,dt = ds \Rightarrow s = \int_0^{20} v\,dt$$

t :	0	2	4	6	8	10	12	14	16	18	20
v :	0	10	18	25	29	32	20	11	5	2	0

$$s = \int_0^{20} v\,dt = \frac{h}{3}\left[(y_0+y_{10})+2(y_2+y_4+y_6+y_8)+4(y_1+y_3+y_5+y_7+y_9)\right]$$

$$= \frac{2}{3}\left[144+320\right] = 309.33 \text{ (Km/min)}$$

Romberg's method:

① Evaluate $\int_0^2 \frac{dx}{x^2+4}$ using Romberg's method. Hence obtain an approximate value for π.

Sol: Let $y = \frac{1}{x^2+4}$ & $I = \int_0^2 \frac{dx}{x^2+4}$

Take $h=1$

The tabulated values of y are

$x:$	0	1	2
$y:$	0.25	0.2	0.125

Using Trapezoidal rule, $I_1 = \int_0^2 \frac{dx}{x^2+4} = \frac{h}{2}[(y_0+y_2)+2y_1] = 0.3875$

Take $h=0.5$. The tabulated values of y are

$x:$	0	0.5	1	1.5	2
$y:$	0.25	0.2353	0.2	0.16	0.125

Using Trapezoidal rule, $I_2 = \int_0^2 \frac{dx}{x^2+4} = \frac{h}{2}[(y_0+y_4)+2(y_1+y_2+y_3)] = 0.3914$

Take $h=0.25$. The tabulated values of y are

$x:$	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2
$y:$	0.25	0.2462	0.2353	0.2192	0.2	0.1798	0.16	0.1416	0.125

By Trapezoidal rule, $I_3 = \frac{h}{2}[(y_0+y_8)+2(y_1+y_2+\cdots+y_7)] = 0.3924$

Using Romberg's formula for I_1 & I_2 we have

$$I = I_2 + \left(\frac{I_2 - I_1}{3}\right) = 0.3927 \quad \text{①}$$

Using Romberg's formula for I_2 & I_3 we have

$$I = I_3 + \left(\frac{I_3 - I_2}{3}\right) = 0.3927 \quad \text{②}$$

Since ① & ② are almost equal we can take $I = \int_0^2 \frac{dx}{x^2+4} = 0.3927$ ③

By actual integration, $\int_0^2 \frac{dx}{x^2+4} = \frac{1}{2}\left[\tan^{-1}(x/2)\right]_0^2 = \frac{1}{2}[\tan^{-1}1 - \tan^{-1}0] = \frac{\pi}{8}$

$= 0.3927$ (∵ by ③)

$\therefore \pi = 3.1416$

② Using Romberg's method evaluate $\int_0^1 \frac{dx}{1+x}$ correct to 4 places of decimals.

Sol: Let $y = \frac{1}{1+x}$ & $I = \int_0^1 \frac{dx}{1+x}$

Take $h=0.5$. The tabulated values of y are

$x:$	0	0.5	1
$y:$	1	0.6667	0.5

Using Trapezoidal rule, $I_1 = \frac{h}{2}[(y_0+y_2)+2(y_1)] = 0.7084$

Take $h=0.25$. The tabulated values of y are

$x:$	0	0.25	0.5	0.75	1
$y:$	1	0.8	0.6667	0.5714	0.5

By Trapezoidal rule, $I_2 = \frac{h}{2}[(y_0+y_4)+2(y_1+y_2+y_3)] = 0.697$

Take h=0.125. The tabulated values of y are

x :	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1
y :	1	0.8889	0.8	0.7273	0.6667	0.6154	0.5714	0.5333	0.5

By Trapezoidal rule, $I_3 = \frac{h}{2}[(y_0+y_8)+2(y_1+y_2+\cdots+y_7)] = 0.6941$

Using Romberg's formula for I_1 & I_2, we have

$I = I_2 + \left(\frac{I_2 - I_1}{3}\right) = 0.6932$ — ①

Using Romberg's formula for I_2 & I_3, we have

$I = I_3 + \left(\frac{I_3 - I_2}{3}\right) = 0.6931$ — ②

Now taking ① & ② values & using Romberg's formula again, we get

$I = 0.6931 + \left(\frac{0.6931 - 0.6932}{3}\right) = 0.6931$

Hence $\int_0^1 \frac{dx}{1+x} = 0.6931$

Two points Gaussian Quadrature Formula:

If the given interval is -1 to 1 then we apply $\int_{-1}^{1} f(x)dx = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$.

Problems:

① Apply Gauss two point formula to evaluate $\int_{-1}^{1} \frac{dx}{1+x^2}$. Also check up the result by actual integration.

Sol: Given interval is -1 to 1 so we apply $\int_{-1}^{1} f(x)dx = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$

Here $f(x) = \frac{1}{1+x^2}$

$f\left(\frac{-1}{\sqrt{3}}\right) = \frac{3}{4}$, $f\left(\frac{1}{\sqrt{3}}\right) = 3/4$

$\therefore \int_{-1}^{1} \frac{dx}{1+x^2} = \frac{3}{4} + \frac{3}{4} = 1.5$

Actual value: $\int_{-1}^{1} \frac{dx}{1+x^2} = [\tan^{-1}x]_{-1}^{1} = \tan^{-1}(1) - \tan^{-1}(-1) = 2\tan^{-1}(1) = 2\left(\frac{\pi}{4}\right)$

$= \frac{\pi}{2} = 1.5708$

Here the error due to two pt/. formula is 0.0708.

② Using Gaussian two pt/. formula evaluate (i) $\int_{-1}^{1} (3x^2+5x^4)dx$ (ii) $\int_{0}^{1} (3x^2+5x^4)dx$

Sol: (i) Given interval is -1 to 1.

Hence we can apply the formula $\int_{-1}^{1} f(x)dx = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$

Here $f(x) = 3x^2+5x^4$

$f\left(\frac{-1}{\sqrt{3}}\right) = \frac{14}{9}$, $f\left(\frac{1}{\sqrt{3}}\right) = \frac{14}{9}$

$\therefore \int_{-1}^{1}(3x^2+5x^4)dx = \frac{14}{9}+\frac{14}{9} = \frac{28}{9} = 3.1111$

(ii) Given interval is 0 to 1, so to make them as -1 to 1.

$\int_{0}^{1}(3x^2+5x^4)dx = \frac{1}{2}\int_{-1}^{1}(3x^2+5x^4)dx$ ($\because 3x^2+5x^4$ is an even fun.)

$= \frac{1}{2}(3.1111) = 1.5556$

Note: If the range for x is (a,b) then by linear transformation, the range (a,b) is mapped into (-1,1).

$x = \left(\frac{b-a}{2}\right)z + \left(\frac{a+b}{2}\right)$, $dx = \left(\frac{b-a}{2}\right)dz$

③ Evaluate $\int_{-2}^{2} e^{-x/2}dx$ by Gauss two pt/. formula.

Sol: The range is not (-1,1), so by using the formula to make them as (-1,1).

$x = \left(\frac{b-a}{2}\right)z + \left(\frac{b+a}{2}\right)$. Here $a=-2$, $b=2$

$x = 2z \Rightarrow dx = 2dz$

$\therefore \int_{-2}^{2} e^{-x/2}dx = \int_{-1}^{1} e^{-z}.2dz = 2\int_{-1}^{1} e^{-z}dz = 2\left[f\left(\frac{-1}{\sqrt{3}}\right)+f\left(\frac{1}{\sqrt{3}}\right)\right]$

Here $f(z) = e^{-z}$

$f\left(\frac{-1}{\sqrt{3}}\right) = 1.7813$, $f\left(\frac{1}{\sqrt{3}}\right) = 0.5614$

$\therefore \int_{-2}^{2} e^{-x/2}dx = 2[1.7813+0.5614] = 4.6854$

④ Using Gaussian two pt formula evaluate $\int_{0}^{\pi/2}\log(1+x)dx$.

Sol: Given range is $(0, \frac{\pi}{2})$. To make them as (-1,1) use

$x = \left(\frac{b-a}{2}\right)z + \left(\frac{b+a}{2}\right)$. Here $a=0$, $b=\frac{\pi}{2}$

$\therefore x = \frac{\pi}{4}z + \frac{\pi}{4} \Rightarrow dx = \frac{\pi}{4}dz$

$\therefore \int_{0}^{\pi/2}\log(1+x)dx = \int_{-1}^{1}\log\left[1+\frac{\pi}{4}z+\frac{\pi}{4}\right]\frac{\pi}{4}dz = \frac{\pi}{4}\int_{-1}^{1}\log\left[1+\frac{\pi}{4}(1+z)\right]dz$

$= \frac{\pi}{4}\left[f\left(\frac{-1}{\sqrt{3}}\right)+f\left(\frac{1}{\sqrt{3}}\right)\right]$

Here $f(z) = \log\left[1+\frac{\pi}{4}(1+z)\right]$. $f\left(\frac{-1}{\sqrt{3}}\right) = 0.2866$, $f\left(\frac{1}{\sqrt{3}}\right) = 0.806$

$\therefore \int_{0}^{\pi/2}\log(1+x)dx = \frac{\pi}{4}[0.2866+0.806] = 0.8581$

Three points Gaussian quadrature formula:

$$\int_{-1}^{1} f(x)\,dx = \frac{5}{9}\left[f\left(-\sqrt{3/5}\right)+f\left(\sqrt{3/5}\right)\right]+\frac{8}{9}f(0)$$

⑤ Using three-point Gaussian quadrature formula, evaluate (i) $\int_{-1}^{1}\frac{dx}{1+x^2}$ (ii) $\int_{0}^{1}\frac{dt}{1+t^2}$.

Sol: (i) Givenrange is $(-1,1)$.

Here $f(x)=\frac{1}{1+x^2}$. $f(0)=1$, $f\left(-\sqrt{3/5}\right)= 5/8$, $f\left(\sqrt{3/5}\right)=\frac{5}{8}$

Three-point Gaussian quadrature formula is

$$\int_{-1}^{1}\frac{dx}{1+x^2} = \frac{5}{9}\left[f\left(-\sqrt{3/5}\right)+f\left(\sqrt{3/5}\right)\right]+\frac{8}{9}f(0) = 1.5833 \text{ — ①}$$

(ii) Here given range is $(0,1)$.

$$\int_{0}^{1}\frac{dt}{1+t^2} = \frac{1}{2}\int_{-1}^{1}\frac{dt}{1+t^2} \quad \left(\because \frac{1}{1+t^2} \text{ is an even fun/.}\right)$$

$$= \frac{1}{2}(1.5833) \quad (\because \text{by ①})$$

$$= 0.7917$$

⑥ Evaluate $\int_{0}^{2}\frac{x^2+2x+1}{1+(x+1)^4}\,dx$ by Gaussian three pt formula.

Sol: Given range is $(0,2)$. To make them as $(-1,1)$.

Let $x=\left(\frac{b-a}{2}\right)z+\left(\frac{b+a}{2}\right)$. Here $a=0$, $b=2$

$\therefore x=z+1 \Rightarrow dx=dz$

$$\int_{0}^{2}\frac{x^2+2x+1}{1+(x+1)^4}\,dx = \int_{-1}^{1}\frac{(z+1)^2+2(z+1)+1}{1+(z+1+1)^4}\,dz = \int_{-1}^{1}\frac{z^2+4z+4}{1+(z+2)^4}\,dz$$

Here $f(z)=\frac{z^2+4z+4}{1+(z+2)^4}$

$f(0)=\frac{4}{17}$, $f\left(-\sqrt{3/5}\right)=0.4613$, $f\left(\sqrt{3/5}\right)=0.1277$

$$\therefore \int_{-1}^{1}\frac{z^2+4z+4}{1+(z+2)^4}\,dz = \frac{5}{9}\left[f\left(-\sqrt{3/5}\right)+f\left(\sqrt{3/5}\right)\right]+\frac{8}{9}f(0)$$

$$= 0.5364$$

⑦ Evaluate $\int_{-1}^{1}\frac{x^2}{1+x^4}\,dx$ by using three pt Gaussian quadrature.

Sol:

Given range is (-1,1).

Gaussian three pt formula is

$$\int_{-1}^{1} f(x)\,dx = \frac{5}{9}\left[f\left(-\sqrt{3/5}\right)+f\left(\sqrt{3/5}\right)\right] + \frac{8}{9}f(0)$$

Here $f(x) = \frac{x^2}{1+x^4}$. $f(0)=0$, $f\left(-\sqrt{3/5}\right) = \frac{15}{34}$, $f\left(\sqrt{3/5}\right) = \frac{15}{34}$

$$\therefore \int_{-1}^{1} \frac{x^2}{1+x^4}\,dx = 0.4902$$

Trapezoidal rule for double integration:

$I = \frac{hk}{4}$ [(Sum of the values of f at the 4 corners)
+2(Sum of the values of f at the remaining nodes on the boundary)
+4(Sum of the values of f at the interior nodes)]

Simpson's rule for double integration:

$I = \frac{hk}{9}$ [(Sum of the values of f at the 4 corners)
+2(Sum of the values of f at the odd positions on the boundary
except the corners)
+4(Sum of the values of f at the even positions on the boundary)
+{4(Sum of the values of f at odd positions)
+8(Sum of the values of f at even positions) on the odd row
of the matrix except boundary rows}
+{8(Sum of the values of f at the odd positions)
+16(Sum of the values of f at the even positions) on the
even rows of the matrix}]

Problems:

① Evaluate $\int_0^1\int_1^2 \frac{2xy}{(1+x^2)(1+y^2)}\,dx\,dy$ by Trapezoidal rule with $h=k=0.25$.

Sol: Here $f(x,y)=\frac{2xy}{(1+x^2)(1+y^2)}$, $h=k=0.25$

y\x	1	1.25	1.5	1.75	2
0	0	0	0	0	0
0.25	0.2353	0.2296	0.2172	0.2027	0.1882
0.5	0.4	0.3902	0.3692	0.3446	0.32
0.75	0.48	0.4683	0.4431	0.4135	0.384
1	0.5	0.4878	0.4615	0.4308	0.4

By Trapezoidal rule,

$$I=\frac{(0.25\times0.25)}{4}\Big[(0+0+0.4+0.5)+2(0+0+0+0.1882+0.32+0.384+0.4308+0.4615+0.4878+0.48+0.4+0.2353)$$
$$+4(0.2296+0.2172+0.2027+0.3902+0.3692+0.3446+0.4683+0.4431+0.4135)\Big]$$

$$=0.3123$$

② Evaluate $\int_1^2\int_1^2 \frac{dx\,dy}{x^2+y^2}$ numerically with $h=0.2$ along x-direction & $k=0.25$ along y-direction.

Sol: Here $f(x,y)=\frac{1}{x^2+y^2}$, $h=0.2$ & $k=0.25$

y\x	1	1.2	1.4	1.6	1.8	2
1	0.5	0.4098	0.3378	0.2809	0.2358	0.2
1.25	0.3902	0.3331	0.2839	0.2426	0.2082	0.1798
1.5	0.3077	0.271	0.2375	0.2079	0.1821	0.16
1.75	0.2462	0.2221	0.1991	0.1779	0.1587	0.1416
2	0.2	0.1838	0.1678	0.1524	0.1381	0.125

By Trapezoidal rule,

$$I=\frac{(0.2\times0.25)}{4}\Big[(0.5+0.2+0.125+0.2)+2(0.4098+0.3378+0.2809+0.2358+0.1798+0.16+0.1416+0.1381+0.1524+0.1678+0.1838+0.2462+0.3077+0.3902)$$
$$+4(0.3331+0.2839+0.2426+0.2082+0.271+0.2375+0.2079+0.1821+0.2221+0.1991+0.1779+0.1587)\Big]=0.2323$$

③ Using Simpson's $\frac{1}{3}^{rd}$ rule evaluate $\int_0^1\int_0^1 \frac{1}{1+x+y}\,dx\,dy$ taking $h=k=0.5$.

Sol: Here $f(x,y) = \frac{1}{1+x+y}$, $h=0.5$ & $k=0.5$

y\x	0	0.5	1
0	1	0.6667	0.5
0.5	0.6667	0.5	0.4
1	0.5	0.4	0.3333

By Simpson's rule,

$$I = \frac{(0.5\times0.5)}{9}\Big[(1+0.5+0.3333+0.5)+2(0)+4(0.6667+0.4+0.4+0.6667)$$
$$+\{4(0)+8(0)\}+\{8(0)+16(0.5)\}\Big]$$

$= 0.5241$

④ The funˢ. $f(x,y)$ is defined by the following table. Compute $\int_1^3\int_0^2 f(x,y)\,dx\,dy$, using Simpson's rule in both direction.

y\x	0	0.5	1	1.5	2
1	2	1.5	1.3	1.4	1.6
2	3.1	2.5	2	2.3	2.9
3	4.2	4	3.8	4.1	4.4

Sol: By Simpson's rule,

$$I = \frac{(0.5\times1)}{9}\Big[(2+1.6+4.4+4.2)+2(1.3+3.8)+4(1.5+1.4+3.1+4+4.1+2.9)$$
$$+\{4(0)+8(0)\}+\{8(2)+16(2.5+2.3)\}\Big]$$

$$= \frac{0.5}{9}\left[12.2+10.2+68+92.8\right]$$

$= 10.1778$

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⑤ Evaluate $\int_{1}^{1.4}\int_{2}^{2.4} \frac{1}{xy}\,dx\,dy$ using Trapezoidal & Simpson's rule verify your result by actual integration.

Sol: Divide the range of x & y into 4 equal parts.

$h = \frac{2.4-2}{4} = 0.1$ & $k = \frac{1.4-1}{4} = 0.1$. Here $f(x,y) = \frac{1}{xy}$.

y\x	2	2.1	2.2	2.3	2.4
1	0.5	0.4762	0.4545	0.4348	0.4167
1.1	0.4545	0.4329	0.4132	0.3953	0.3788
1.2	0.4167	0.3968	0.3788	0.3623	0.3472
1.3	0.3846	0.3663	0.3497	0.3344	0.3205
1.4	0.3571	0.3401	0.3247	0.3106	0.2976

By Trapezoidal rule,

$$I = \frac{(0.1\times0.1)}{4}\Big[(0.5+0.4167+0.2976+0.3571)+2(0.4762+0.4545+0.3472+0.3205+0.3106+0.3247+$$
$$+0.4348+0.3788+0.3472+0.3205+0.3106+0.3247+$$
$$0.3401+0.3846+0.4167+0.4545)$$
$$+4(0.4329+0.4132+0.3953+0.3968+0.3788+0.3623$$
$$+0.3663+0.3497+0.3344)\Big]$$

$= 0.0614$

By Simpson's rule,

$$I = \frac{(0.1\times0.1)}{9}\Big[(0.5+0.4167+0.2976+0.3571)+2(0.4167+0.3472+0.4545$$
$$+0.3247)+4(0.4545+0.3846+0.4762+0.4348+0.3788$$
$$+0.3205+0.3401+0.3106)+\{4(0.3788)+8(0.3968+0.3623)\}$$
$$+\{8(0.4132+0.3497)+16(0.4329+0.3953+0.3663+0.3344\}$$

$= 0.0613$

Actual Value:

$$\int_{1}^{1.4}\int_{2}^{2.4}\frac{1}{xy}\,dx\,dy = \int_{1}^{1.4}\frac{1}{y}(\log x)_{2}^{2.4}\,dy = \int_{1}^{1.4}\frac{1}{y}(\log 2.4-\log 2)\,dy$$
$$= (\log 2.4-\log 2)(\log 1.4-\log 1) = \log(1.2)\log(1.4)$$
$$= 0.0613$$

INITIAL VALUE PROBLEMS FOR ORDINARY DIFFERENTIAL EQUATIONS

Defns:

An eqn. which involves the differential coeffs. of one variable, is called the ordinary differential eqn..

Single step methods (or) Pointwise methods:

A series for y interms of powers of x, from which the value of y can be obtained by direct substitution. The methods of Taylor & Picard belong to this type. In these methods y is approximated by a truncated series & each term of the series is a fun. of x. The information about the curve at one pt. is used & the sol. is not iterated. Hence these methods are called single step methods or Pointwise methods.

Multistep methods (or) Step by step methods:

In a set of tabulated values of x & y, we obtain y by iterative process. The methods of Euler, Range-kutta, Milne, Adam, etc., belong to this type. Here the values of y are computed by short steps for equal intervals h of the independent variable. These values are iterated till we get the desired accuracy. Hence these methods are called step by step methods.

Taylor Series method:

Taylor series expansion of $y(x)$ at $x=x_0$ is given by

$$y(x)=y(x_0)+(x-x_0)y'(x_0)+\frac{(x-x_0)^2}{2!}y''(x_0)+\cdots$$

$$=y_0+(x-x_0)y_0'+\frac{(x-x_0)^2}{2!}y_0''+\cdots$$

Let $x_1=x_0+h$

$$y(x_1)=y_1=y_0+\frac{h}{1!}y_0'+\frac{h^2}{2!}y_0''+\cdots$$

Let $x_2=x_1+h$

$$y(x_1+h)=y(x_2)=y_2=y_1+\frac{h}{1!}y_1'+\frac{h^2}{2!}y_1''+\cdots$$

Continuing in this way we find the sol. $y(x)$.

Problems:

① Using Taylor series method find y at $x=0.1$ if $\frac{dy}{dx}=x^2y-1$, $y(0)=1$.

Sol:

Given $y'=x^2y-1$ & $x_0=0$, $y_0=1$, $h=0.1$

$y'=x^2y-1$ $\qquad y_0'=-1$

$y''=x^2y'+2xy$ $\qquad y_0''=0$

$y'''=x^2y''+4xy'+2y$ $\qquad y_0'''=2$

$y^{IV}=x^2y'''+6xy''+6y'$ $\qquad y_0^{IV}=-6$

Taylor series formula: $y(x)=y_0+(x-x_0)y_0'+\frac{(x-x_0)^2}{2!}y_0''+\frac{(x-x_0)^3}{3!}y_0'''+\frac{(x-x_0)^4}{4!}y_0^{IV}+\cdots$

$$y(x)=1+x(-1)+\frac{x^2}{2}(0)+\frac{x^3}{6}(2)+\frac{x^4}{24}(-6)+\cdots$$

$$=1-x+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$$

$\therefore y(0.1)=0.9003$

---

② Solve $y'=x+y$; $y(0)=1$ by Taylor series method. Find the values of y at $x=0.1$ & $x=0.2$.

Sol:

Given $y'=x+y$, $x_0=0$, $y_0=1$, $h=0.1$

$y'=x+y$ $\qquad y_0'=1$

$y''=1+y'$ $\qquad y_0''=2$

$y'''=y''$ $\qquad y_0'''=2$

$y^{IV}=y'''$ $\qquad y_0^{IV}=2$

Taylor series formula: $y(x)=y_0+(x-x_0)y_0'+\frac{(x-x_0)^2}{2!}y_0''+\frac{(x-x_0)^3}{3!}y_0'''+\frac{(x-x_0)^4}{4!}y_0^{IV}+\cdots$

$$y(x)=1+x(1)+\frac{x^2}{2}(2)+\frac{x^3}{6}(2)+\frac{x^4}{24}(2)+\cdots$$

$$=1+x+x^2+\frac{x^3}{3}+\frac{x^4}{12}+\cdots$$

$\therefore y(0.1)=1.1103$ & $y(0.2)=1.2428$

---

③ Solve $\frac{dy}{dx}=y^2+x^2$ with $y(0)=1$. Use Taylor series at $x=0.1, 0.2$ & 0.4.

Sol:

Given $y'=y^2+x^2$, $x_0=0$, $y_0=1$, $h=0.1$

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$y' = y^2 + x^2$ $\qquad$ $y_0' = 1$

$y'' = 2yy' + 2x$ $\qquad$ $y_0'' = 2$

$y''' = 2yy'' + 2y'^2 + 2$ $\qquad$ $y_0''' = 8$

$y^{iv} = 2yy''' + 6y'y''$ $\qquad$ $y_0^{iv} = 28$

Taylor series formula: $y(x) = y_0 + (x-x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \frac{(x-x_0)^3}{3!}y_0''' + \frac{(x-x_0)^4}{4!}y_0^{iv} + \cdots$

$$y(x) = 1 + x(1) + \frac{x^2}{2}(2) + \frac{x^3}{6}(8) + \frac{x^4}{24}(28) + \cdots$$

$$= 1 + x + x^2 + \frac{4x^3}{3} + \frac{7x^4}{6} + \cdots$$

$y(0.1) = 1.1114$, $y(0.2) = 1.2525$, $y(0.4) = 1.6752$

④ Find the Taylor series solution with 4 terms for the initial value problem, $\frac{dy}{dx} = x^3 + y$, $y(1) = 1$.

Sol:

Given $y' = x^3 + y$, $x_0 = 1$, $y_0 = 1$

$y' = x^3 + y$ $\qquad$ $y_0' = 2$

$y'' = 3x^2 + y'$ $\qquad$ $y_0'' = 5$

$y''' = 6x + y''$ $\qquad$ $y_0''' = 11$

$y^{iv} = 6 + y'''$ $\qquad$ $y_0^{iv} = 17$

Taylor series formula: $y(x) = y_0 + (x-x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \frac{(x-x_0)^3}{3!}y_0''' + \frac{(x-x_0)^4}{4!}y_0^{iv} + \cdots$

$$y(x) = 1 + (x-1)(2) + \frac{(x-1)^2}{2}(5) + \frac{(x-1)^3}{6}(11) + \frac{(x-1)^4}{24}(17) + \cdots$$

$$= 1 + 2(x-1) + \frac{5(x-1)^2}{2} + \frac{11(x-1)^3}{6} + \frac{17(x-1)^4}{24} + \cdots$$

⑤ Using Taylor series method with the first 5 terms in the expansion find $y(0.1)$ correct to 3 decimal places, given that $\frac{dy}{dx} = e^x - y^2$, $y(0) = 1$.

Sol:

Given $y' = e^x - y^2$, $x_0 = 0$, $y_0 = 1$

$y' = e^x - y^2$ $\qquad$ $y_0' = 0$

$y'' = e^x - 2yy'$ $\qquad$ $y_0'' = 1$

$y''' = e^x - 2yy'' - 2y'^2$ $\qquad y_0''' = -1$

$y^{IV} = e^x - 2yy''' - 6y'y''$ $\qquad y_0^{IV} = 3$

$y^{V} = e^x - 2yy^{IV} - 8y'y''' - 6y''^2$ $\qquad y_0^{V} = -11$

Taylor series formula: $y = y_0 + (x-x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \frac{(x-x_0)^3}{3!}y_0''' + \frac{(x-x_0)^4}{4!}y_0^{IV} + \frac{(x-x_0)^5}{5!}y_0^{V} + \cdots$

$$y(x) = 1 + x(0) + \frac{x^2}{2}(1) + \frac{x^3}{6}(-1) + \frac{x^4}{24}(3) + \frac{x^5}{120}(-11) + \cdots$$

$$= 1 + x^2 - \frac{x^3}{6} + \frac{x^4}{8} - \frac{11x^5}{120} + \cdots$$

$\therefore y(0.1) = 1.0048$

Taylor series method for simultaneous first order differential eqns.:

⑥ Solve the system of eqns. $\frac{dy}{dx} = z - x^2$, $\frac{dz}{dx} = y + x$ with $y(0) = 1$, $z(0) = 1$ by taking $h = 0.1$, to get $y(0.1)$ & $z(0.1)$.

Sol:

Given $y' = z - x^2$, $z' = y + x$, $x_0 = 0, y_0 = 1, z_0 = 1$

$y' = z - x^2$ $\qquad y_0' = 1$ $\qquad z' = y + x$ $\qquad z_0' = 1$

$y'' = z' - 2x$ $\qquad y_0'' = 1$ $\qquad z'' = y' + 1$ $\qquad z_0'' = 2$

$y''' = z'' - 2$ $\qquad y_0''' = 0$ $\qquad z''' = y''$ $\qquad z_0''' = 1$

$y^{IV} = z'''$ $\qquad y_0^{IV} = 1$ $\qquad z^{IV} = y'''$ $\qquad z_0^{IV} = 0$

Taylor series formula:

$$y(x) = y_0 + (x-x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \frac{(x-x_0)^3}{3!}y_0''' + \frac{(x-x_0)^4}{4!}y_0^{IV} + \cdots$$

$$= 1 + x(1) + \frac{x^2}{2}(1) + \frac{x^3}{6}(0) + \frac{x^4}{24}(1) + \cdots$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^4}{24} + \cdots$$

$\therefore y(0.1) = 1.105$

$$z(x) = z_0 + (x-x_0)z_0' + \frac{(x-x_0)^2}{2!}z_0'' + \frac{(x-x_0)^3}{3!}z_0''' + \frac{(x-x_0)^4}{4!}z_0^{IV} + \cdots$$

$$= 1 + x(1) + \frac{x^2}{2}(2) + \frac{x^3}{6}(1) + \frac{x^4}{24}(0) + \cdots$$

$$= 1 + x + x^2 + \frac{x^3}{6} + \cdots$$

$\therefore z(0.1) = 1.1102$

Solving Higher order linear differential eqns). by Taylor series method:

⑦ By Taylor series method find $y(0.1)$ given that $y''=y+xy'$, $y(0)=1$, $y'(0)=0$.

Sol:

Given $y''=y+xy'$, $x_0=0, y_0=1, y_0'=0$

$$y''=y+xy' \qquad y_0''=1$$

$$y'''=y'+xy''+y'=xy''+2y' \qquad y_0'''=0$$

$$y^{iv}=3y''+xy''' \qquad y_0^{iv}=3$$

$$y^{v}=4y'''+xy^{iv} \qquad y_0^{v}=0$$

Taylor Series formula: $y(x)=y_0+(x-x_0)y_0'+\frac{(x-x_0)^2}{2!}y_0''+\frac{(x-x_0)^3}{3!}y_0'''+\frac{(x-x_0)^4}{4!}y_0^{iv}+\frac{(x-x_0)^5}{5!}y_0^{v}+\cdots$

$$y(x)=1+x(0)+\frac{x^2}{2}(1)+\frac{x^3}{6}(0)+\frac{x^4}{24}(3)+\frac{x^5}{120}(0)+\cdots$$

$$=1+\frac{x^2}{2}+\frac{x^4}{8}+\cdots$$

$\therefore y(0.1)=1.005$

---

⑧ Evaluate the values of $y(0.1)$ & $y(0.2)$ given $y''-x(y')^2+y^2=0$, $y(0)=1$, $y'(0)=0$ by using Taylor series method.

Sol:

Given $y''=x(y')^2-y^2$, $x_0=0, y_0=1, y_0'=0$

$$y''=x(y')^2-y^2 \qquad y_0''=-1$$

$$y'''=2xy'y''+(y')^2-2yy' \qquad y_0'''=0$$

$$y^{iv}=2xy'y'''+2x(y'')^2+4y'y''-2yy''-2y'^2 \qquad y_0^{iv}=2$$

Taylor series formula: $y=y_0+(x-x_0)y_0'+\frac{(x-x_0)^2}{2!}y_0''+\frac{(x-x_0)^3}{3!}y_0'''+\frac{(x-x_0)^4}{4!}y_0^{iv}+\frac{(x-x_0)^5}{5!}y_0^{v}+\cdots$

$$y(x)=1+x(0)+\frac{x^2}{2}(-1)+\frac{x^3}{6}(0)+\frac{x^4}{24}(2)+\cdots$$

$$=1-\frac{x^2}{2}+\frac{x^4}{12}+\cdots$$

$\therefore y(0.1)=0.995$, $y(0.2)=0.9801$

Euler's method (or) Euler's Algorithm:

$$y_{n+1} = y_n + h f(x_n, y_n) \quad , n=0,1,2,\ldots$$

Modified Euler's method:

$$y_{n+1} = y_n + h\left[f\left(x_n + \frac{h}{2},\ y_n + \frac{h}{2} f(x_n, y_n)\right)\right] \quad , n=0,1,2,\ldots$$

Problems:

① Using Euler's method find $y(0.2), y(0.4)$ & $y(0.6)$ from $\frac{dy}{dx} = x+y$, $y(0)=1$ with $h=0.2$.

Sol:

Given $\frac{dy}{dx} = f(x,y) = x+y$, $x_0=0, y_0=1, h=0.2, x_1=0.2, x_2=0.4, x_3=0.6$

By Euler's algorithm, $y_{n+1} = y_n + h f(x_n, y_n)$, $n=0,1,2,\ldots$

$y_1 = y_0 + h f(x_0, y_0)$

$= 1 + (0.2)[x_0 + y_0]$

$= 1.2$

$y_2 = y_1 + h f(x_1, y_1) = 1.2 + (0.2)[0.2 + 1.2] = 1.48$

$y_3 = y_2 + h f(x_2, y_2) = 1.48 + (0.2)[0.4 + 1.48] = 1.856$

Hence $y(0.2) = 1.2$, $y(0.4) = 1.48$ & $y(0.6) = 1.856$

② Using Euler's method solve $y' = x+y+xy$, $y(0)=1$ compute y at $x=0.1$, by taking $h=0.05$.

Sol:

Given $f(x,y) = x+y+xy$, $x_0=0, y_0=1, h=0.05, x_1=0.05, x_2=0.1$

Euler's formula: $y_{n+1} = y_n + h f(x_n, y_n)$, $n=0,1,2,\ldots$

$y_1 = y_0 + h f(x_0, y_0) = 1 + 0.05(1) = 1.05$

$\therefore y(0.05) = 1.05$

$y_2 = y_1 + h f(x_1, y_1) = 1.05 + 0.05(1.1525) = 1.1076$

$\therefore y(0.1) = 1.1076$

③ Using Euler's method find $y(0.3)$ of $y(x)$ satisfies the initial value problem, $\frac{dy}{dx} = \frac{1}{2}(x^2+1)y^2$, $y(0.2)=1.1114$.

Sol:

Given $f(x,y)=\frac{1}{2}(x^2+1)y^2$, $x_0=0.2$, $y_0=1.1114$, $h=0.1$, $x_1=0.3$

Euler's formula: $y_{n+1}=y_n+hf(x_n,y_n)$, $n=0,1,2,\ldots$

$y_1=y_0+hf(x_0,y_0)=1.1114+(0.1)\left[\frac{1}{2}(0.2^2+1)\,(1.1114)^2\right]=1.1756$

$\therefore y(0.3)=1.1756$

④ Using Modified Euler's method, compute $y(0.1)$ with $h=0.1$ from $y'=y-\frac{2x}{y}$, $y(0)=1$.

Sol:

Given $f(x,y)=y-\frac{2x}{y}$, $x_0=0$, $y_0=1$, $h=0.1$, $x_1=0.1$

Modified Euler's formula: $y_{n+1}=y_n+h\left[f\left(x_n+\frac{h}{2}, y_n+\frac{h}{2}f(x_n,y_n)\right)\right]$, $n=0,1,2,\ldots$

$y_1=y_0+h\left[f\left(x_0+\frac{h}{2}, y_0+\frac{h}{2}f(x_0,y_0)\right)\right]$

$=1+0.1\left[f\left(0+\frac{0.1}{2}, 1+\frac{0.1}{2}f(0,1)\right)\right]$

$=1+0.1f(0.05,1.05)=1+0.1[1.05-0.0952]$

$=1.0955$

$\therefore y(0.1)=1.0955$

⑤ Solve $y'=1-y$, $y(0)=0$ by modified Euler's method.

Sol:

Given $f(x,y)=1-y$, $x_0=0$, $y_0=0$, $h=0.1$, $x_1=0.1$, $x_2=0.2$, $x_3=0.3$

Modified Euler's method: $y_{n+1}=y_n+h\left[f\left(x_n+\frac{h}{2}, y_n+\frac{h}{2}f(x_n,y_n)\right)\right]$, $n=0,1,2,\ldots$

$y_1=y_0+h\left[f\left(x_0+\frac{h}{2}, y_0+\frac{h}{2}f(x_0,y_0)\right)\right]$

$=0.1f(0.05,0.05)=0.095$

$y_2=y_1+h\left[f\left(x_1+\frac{h}{2}, y_1+\frac{h}{2}f(x_1,y_1)\right)\right]$

$=0.095+0.1f(0.15,0.1403)=0.181$

$y_3=y_2+h\left[f\left(x_2+\frac{h}{2}, y_2+\frac{h}{2}f(x_2,y_2)\right)\right]$

$=0.181+0.1f(0.25,0.2219)=0.2588$

Hence $y(0.1)=0.095$, $y(0.2)=0.181$ & $y(0.3)=0.2588$

⑥ Consider the initial value problem $\frac{dy}{dx}=y-x^2+1$, $y(0)=0.5$ using the modified Euler's method, find $y(0.2)$.

Sol:

Given $f(x,y)=y-x^2+1$, $x_0=0$, $y_0=0.5$, $h=0.2$, $x_1=0.2$

Modified Euler's formula: $y_{n+1}=y_n+h\left[f\left(x_n+\frac{h}{2}, y_n+\frac{h}{2}f(x_n,y_n)\right)\right]$, $n=0,1,2,\dots$

$y_1=y_0+h\left[f\left(x_0+\frac{h}{2}, y_0+\frac{h}{2}f(x_0,y_0)\right)\right]$

$=0.5+0.2\,f(0.1, 0.65)$

$=0.5+0.2(1.64)=0.828$

$\therefore y(0.2)=0.828$

Fourth order Runge-Kutta method for solving first & second order eqns/.:

Second order R-K method:

$k_1=hf(x,y)$

$k_2=hf\left[x+\frac{h}{2}, y+\frac{k_1}{2}\right]$

$\Delta y=k_2$ where $h=\Delta x$

Third order R-K method:

$k_1=hf(x,y)$

$k_2=hf\left[x+\frac{h}{2}, y+\frac{k_1}{2}\right]$

$k_3=hf\left[x+h, y+2k_2-k_1\right]$ & $\Delta y=\frac{1}{6}(k_1+4k_2+k_3)$

Fourth order R-K method:

$k_1=hf(x,y)$

$k_2=hf\left[x+\frac{h}{2}, y+\frac{k_1}{2}\right]$

$k_3=hf\left[x+\frac{h}{2}, y+\frac{k_2}{2}\right]$

$k_4=hf\left[x+h, y+k_3\right]$

& $\Delta y=\frac{1}{6}(k_1+2k_2+2k_3+k_4)$

Problems:

① Given $\frac{dy}{dx}=x^3+y$, $y(0)=2$. Compute $y(0.2)$, $y(0.4)$ & $y(0.6)$ by R-K method of 4th order.

Sol:

Given $y'=f(x,y)=x^3+y$, $x_0=0$, $y_0=2$, $h=0.2$, $x_1=0.2$, $x_2=0.4$, $x_3=0.6$.

Fourth order R-K formula: $k_1=hf(x,y)$; $k_2=hf\left[x+\frac{h}{2}, y+\frac{k_1}{2}\right]$;

$k_3 = hf[x+\frac{h}{2}, y+\frac{k_2}{2}]$; $k_4 = hf[x+h, y+k_3]$; $\Delta y = \frac{1}{6}(k_1+2k_2+2k_3+k_4)$

1st interval:

$k_1 = hf(x_0, y_0) = 0.4$

$k_2 = hf[x_0+\frac{h}{2}, y_0+\frac{k_1}{2}] = (0.2)f(0.1, 2.2) = 0.4402$

$k_3 = hf[x_0+\frac{h}{2}, y_0+\frac{k_2}{2}] = (0.2)f(0.1, 2.2201) = 0.4442$

$k_4 = hf[x_0+h, y_0+k_3] = (0.2)f(0.2, 2.4442) = 0.4904$

$\Delta y = \frac{1}{6}(k_1+2k_2+2k_3+k_4) = 0.4432$

$\therefore y(0.2) = y_1 = y_0 + \Delta y = 2.4432$

2nd interval:

$k_1 = hf(x_1, y_1) = 0.4902$

$k_2 = hf[x_1+\frac{h}{2}, y_1+\frac{k_1}{2}] = (0.2)f(0.3, 2.6883) = 0.5431$

$k_3 = hf[x_1+\frac{h}{2}, y_1+\frac{k_2}{2}] = (0.2)f(0.3, 2.7148) = 0.5484$

$k_4 = hf[x_1+h, y_1+k_3] = (0.2)f(0.4, 2.9916) = 0.6111$

$\Delta y = \frac{1}{6}(k_1+2k_2+2k_3+k_4) = 0.5474$

$\therefore y(0.4) = y_2 = y_1 + \Delta y = 2.9906$

3rd interval:

$k_1 = hf(x_2, y_2) = 0.6109$

$k_2 = hf[x_2+\frac{h}{2}, y_2+\frac{k_1}{2}] = (0.2)f(0.5, 3.2961) = 0.6842$

$k_3 = hf[x_2+\frac{h}{2}, y_2+\frac{k_2}{2}] = (0.2)f(0.5, 3.3327) = 0.6915$

$k_4 = hf[x_2+h, y_2+k_3] = (0.2)f(0.6, 3.6821) = 0.7796$

$\Delta y = \frac{1}{6}(k_1+2k_2+2k_3+k_4) = 0.6903$

$\therefore y(0.6) = y_3 = y_2 + \Delta y = 3.6809$

② Using R-K method of 4th order, solve $\frac{dy}{dx} = \frac{y^2-x^2}{y^2+x^2}$ with $y(0)=1$ at $x=0.2$.

Sol: Given $\frac{dy}{dx} = f(x,y) = \frac{y^2-x^2}{y^2+x^2}$, $x_0 = 0, y_0 = 1, h = 0.2, x_1 = 0.2$

$k_1 = hf(x_0, y_0) = (0.2)f(0, 1) = 0.2$

$k_2 = hf[x_0+\frac{h}{2}, y_0+\frac{k_1}{2}] = (0.2)f(0.1, 1.1) = 0.1967$

$k_3 = hf[x_0+\frac{h}{2}, y_0+\frac{k_2}{2}] = (0.2)f(0.1, 1.0984) = 0.1967$

$k_4 = hf[x_0+h, y_0+k_3] = (0.2)f(0.2, 1.1967) = 0.1891$

$\Delta y = \frac{1}{6}(k_1+2k_2+2k_3+k_4) = 0.196$

$\therefore y(0.2) = y_1 = y_0 + \Delta y = 1.196$

③ Find $y(0.8)$ given that $y' = y - x^2$, $y(0.6) = 1.7379$ by using R-K method of 4th order. Take $h = 0.1$

Sol: Given $y' = f(x,y) = y - x^2$, $x_0 = 0.6$, $y_0 = 1.7379$, $h = 0.1$, $x_1 = 0.7$, $x_2 = 0.8$

First interval:

$k_1 = hf(x_0, y_0) = (0.1)f(0.6, 1.7379) = 0.1378$

$k_2 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right] = (0.1)f(0.65, 1.8068) = 0.1384$

$k_3 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right] = (0.1)f(0.65, 1.8071) = 0.1385$

$k_4 = hf[x_0 + h, y_0 + k_3] = (0.1)f(0.7, 1.8764) = 0.1386$

$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.1384$

$\therefore y(0.7) = y_1 = y_0 + \Delta y = 1.8763$

Second interval:

$k_1 = hf(x_1, y_1) = (0.1)f(0.7, 1.8763) = 0.1386$

$k_2 = hf\left[x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right] = (0.1)f(0.75, 1.9456) = 0.1383$

$k_3 = hf\left[x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right] = (0.1)f(0.75, 1.9455) = 0.1383$

$k_4 = hf[x_1 + h, y_1 + k_3] = (0.1)f(0.8, 2.0146) = 0.1375$

$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.1382$

$\therefore y(0.8) = y_2 = y_1 + \Delta y = 2.0145$

R-K method for simultaneous first order differential eqn.:

Solving the system of differential eqns. $\frac{dy}{dx} = f_1(x,y,z)$ & $\frac{dz}{dx} = f_2(x,y,z)$ with the initial conditions $y(x_0) = y_0$, $z(x_0) = z_0$. [Here x is independent variable while y & z are dependent variable]

$k_1 = hf_1(x_0, y_0, z_0)$

$k_2 = hf_1\left[x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right]$

$k_3 = hf_1\left[x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right]$

$k_4 = hf_1[x_0 + h, y_0 + k_3, z_0 + l_3]$

$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$y_1 = y_0 + \Delta y$

$l_1 = hf_2(x_0, y_0, z_0)$

$l_2 = hf_2\left[x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right]$

$l_3 = hf_2\left[x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right]$

$l_4 = hf_2[x_0 + h, y_0 + k_3, z_0 + l_3]$

$\Delta z = \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$

$z_1 = z_0 + \Delta z$

④ Solving the system of differential equn/. $\frac{dy}{dx} = xz+1$, $\frac{dz}{dx} = -xy$ for $x=0.3$ using 4th order R-K method, the initial values are $x=0, y=0, z=1$.

Sol: Given $f_1(x,y,z) = \frac{dy}{dx} = xz+1$, $f_2(x,y,z) = \frac{dz}{dx} = -xy$, $x_0=0, y_0=0, z_0=1$,

$h=0.3$, $x_1=0.3$

$k_1 = hf_1(x_0,y_0,z_0)$

$= (0.3)f_1(0,0,1) = 0.3$

$k_2 = hf_1[x_0+\frac{h}{2}, y_0+\frac{k_1}{2}, z_0+\frac{l_1}{2}]$

$= (0.3)f_1(0.15, 0.15, 1) = 0.345$

$k_3 = hf_1[x_0+\frac{h}{2}, y_0+\frac{k_2}{2}, z_0+\frac{l_2}{2}]$

$= (0.3)f_1(0.15, 0.1725, 0.9966) = 0.3448$

$k_4 = hf_1[x_0+h, y_0+k_3, z_0+l_3]$

$= (0.3)f_1(0.3, 0.3448, 0.9922) = 0.3893$

$\Delta y = \frac{1}{6}(k_1+2k_2+2k_3+k_4) = 0.3448$

$y_1 = y_0 + \Delta y = 0.3448$

$l_1 = hf_2(x_0,y_0,z_0)$

$= (0.3)f_2(0,0,1) = 0$

$l_2 = hf_2[x_0+\frac{h}{2}, y_0+\frac{k_1}{2}, z_0+\frac{l_1}{2}]$

$= (0.3)f_2(0.15, 0.15, 1) = -0.0068$

$l_3 = hf_2[x_0+\frac{h}{2}, y_0+\frac{k_2}{2}, z_0+\frac{l_2}{2}]$

$= (0.3)f_2(0.15, 0.1725, 0.9966) = -0.0078$

$l_4 = hf_2[x_0+h, y_0+k_3, z_0+l_3]$

$= (0.3)f_2(0.3, 0.3448, 0.9922) = -0.031$

$\Delta z = \frac{1}{6}(l_1+2l_2+2l_3+l_4) = -0.01$

$z_1 = z_0 + \Delta z = 0.99$

R-K method for second order differential eqns/.:

⑤ Consider the second order initial value problem $y''-2y'+2y = e^{2t}\sin t$ with $y(0) = -0.4$ & $y'(0) = -0.6$ using 4th order R-K method, find $y(0.2)$.

Sol: Let $t = x$

$y'' = 2y' - 2y + e^{2x}\sin x$, $x_0 = 0$, $y_0 = -0.4$, $y_0' = -0.6$, $h = 0.2$, $x_1 = 0.2$

Setting $y' = z$ the eqn/. becomes, $z' = 2z - 2y + e^{2x}\sin x$

$f_1(x,y,z) = \frac{dy}{dx} = z$

$f_2(x,y,z) = \frac{dz}{dx} = 2z - 2y + e^{2x}\sin x$

Given $x_0 = 0$, $y_0 = -0.4$, $y_0' = z_0 = -0.6$, $h = 0.2$, $x_1 = 0.2$

$k_1 = hf_1(x_0,y_0,z_0)$

$= (0.2)f_1(0, -0.4, -0.6) = -0.12$

$k_2 = hf_1[x_0+\frac{h}{2}, y_0+\frac{k_1}{2}, z_0+\frac{l_1}{2}]$

$= (0.2)f_1(0.1, -0.46, -0.64) = -0.128$

$k_3 = hf_1[x_0+\frac{h}{2}, y_0+\frac{k_2}{2}, z_0+\frac{l_2}{2}]$

$= (0.2)f_1(0.1, -0.464, -0.6238) = -0.1248$

$l_1 = hf_2(x_0,y_0,z_0)$

$= (0.2)f_2(0, -0.4, -0.6) = -0.08$

$l_2 = hf_2[x_0+\frac{h}{2}, y_0+\frac{k_1}{2}, z_0+\frac{l_1}{2}]$

$= (0.2)f_2(0.1, -0.46, -0.64) = -0.0476$

$l_3 = hf_2[x_0+\frac{h}{2}, y_0+\frac{k_2}{2}, z_0+\frac{l_2}{2}]$

$= (0.2)f_2(0.1, -0.464, -0.6238)$

$= -0.0395$

$k_4 = hf_1[x_0+h,\ y_0+k_3,\ z_0+l_3]$

$= (0.2) f_1(0.2, -0.5248, -0.6395) = -0.1279$

$l_4 = hf_2[x_0+h,\ y_0+k_3,\ z_0+l_3]$

$= (0.2) f_2(0.2, -0.5248, -0.6395)$

$= 0.0134$

$\Delta y = \frac{1}{6}(k_1+2k_2+2k_3+k_4) = -0.1256$

$\Delta z = \frac{1}{6}(l_1+2l_2+2l_3+l_4) = 0.0134$

$y_1 = y(0.2) = y_0 + \Delta y = -0.5256$

Milne's Predictor & Corrector methods:

In the Milne's method, we suppose that 4 equispaced starting values of y are known, at the pts x_n, x_{n-1}, x_{n-2} & x_{n-3}.

Milne's predictor & corrector formula is

$$y_{n+1,p} = y_{n-3} + \frac{4h}{3}[2y'_{n-2} - y'_{n-1} + 2y'_n]$$

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}[y'_{n-1} + 4y'_n + y'_{n+1}]$$

Problems:

① Given $\frac{dy}{dx} = x^3+y$, $y(0)=2$. The values of $y(0.2)=2.073$, $y(0.4)=2.452$ & $y(0.6)=3.023$ are got by R-K method of 4th order. Find $y(0.8)$ by Milne's predictor corrector method taking $h=0.2$.

Sol: Here $x_0=0$, $x_1=0.2$, $x_2=0.4$, $x_3=0.6$, $h=0.2$

$y_0=2$, $y_1=2.073$, $y_2=2.452$, $y_3=3.023$

$y' = x^3+y$

Milne's predictor formula: $y_{n+1,p} = y_{n-3} + \frac{4h}{3}[2y'_{n-2} - y'_{n-1} + 2y'_n]$

$y_{4,p} = y_0 + \frac{4h}{3}[2y'_1 - y'_2 + 2y'_3]$

$y'_1 = x_1^3 + y_1 = 2.081$, $y'_2 = x_2^3+y_2 = 2.516$, $y'_3 = x_3^3+y_3 = 3.239$

$\therefore y_{4,p} = 2 + \frac{4(0.2)}{3}[2(2.081) - 2.516 + 2(3.239)] = 4.1664$

Milne's corrector formula: $y_{n+1,c} = y_{n-1} + \frac{h}{3}[y'_{n-1} + 4y'_n + y'_{n+1}]$

$y_{4,c} = y_2 + \frac{h}{3}[y'_2 + 4y'_3 + y'_4]$

$y'_4 = x_4^3 + y_4 = 4.6784$

$\therefore y_{4,c} = 2.452 + \frac{0.2}{3}[2.516 + 4(3.239) + 4.6784] = 3.7954$

$\therefore y(0.8) = 3.7954$

---

② Solve $y'=x-y^2$, $0\le x\le 1$, $y(0)=0$, $y(0.2)=0.02$, $y(0.4)=0.0795$, $y(0.6)=0.1762$ by Milne's method to find $y(0.8)$ & $y(1)$.

Sol: Here $x_0=0$, $x_1=0.2$, $x_2=0.4$, $x_3=0.6$, $x_4=0.8$, $x_5=1$, $h=0.2$

$y_0=0$, $y_1=0.02$, $y_2=0.0795$, $y_3=0.1762$

Given $y'=x-y^2$

Milne's predictor formula: $y_{n+1,p}=y_{n-3}+\frac{4h}{3}[2y'_{n-2}-y'_{n-1}+2y'_n]$

$y_{4,p}=y_0+\frac{4h}{3}[2y_1'-y_2'+2y_3']$

$y_1'=x_1-y_1^2=0.1996$, $y_2'=x_2-y_2^2=0.3937$, $y_3'=x_3-y_3^2=0.569$

$\therefore y_{4,p}=\frac{4(0.2)}{3}[2(0.1996)-0.3937+2(0.569)]=0.3049$

Milne's corrector formula: $y_{n+1,c}=y_{n-1}+\frac{h}{3}[y'_{n-1}+4y'_n+y'_{n+1}]$

$y_{4,c}=y_2+\frac{h}{3}[y_2'+4y_3'+y_4']$

$y_4'=x_4-y_4^2=0.707$

$\therefore y_{4,c}=0.0795+\frac{0.2}{3}[0.3937+4(0.569)+0.707]=0.3046$

$\therefore y(0.8)=0.3046=y_4$

Milne's predictor formula:

$y_{5,p}=y_1+\frac{4h}{3}[2y_2'-y_3'+2y_4']$

$y_4'=x_4-y_4^2=0.7072$

$\therefore y_{5,p}=0.02+\frac{4(0.2)}{3}[2(0.3937)-0.569+2(0.7072)]=0.4554$

Milne's corrector formula:

$y_{5,c}=y_3+\frac{h}{3}[y_3'+4y_4'+y_5']$

$y_5'=x_5-y_5^2=0.7926$

$\therefore y_{5,c}=0.1762+\frac{0.2}{3}[0.569+4(0.7072)+0.7926]=0.4556$

$\therefore y(1)=0.4556$

---

③ Given $y'=1-y$ & $y(0)=0$, find (i) $y(0.1)$ by Euler's method (ii) $y(0.2)$ by Modified Euler's method (iii) $y(0.4)$ by Milne's method.

Sol: Given $y'=1-y$, $x_0=0$, $y_0=0$, $h=0.1$, $x_1=0.1$, $x_2=0.2$, $x_3=0.3$, $x_4=0.4$

(i) Euler's method: $y_{n+1}=y_n+hf(x_n,y_n)$

$y_1=y_0+hf(x_0,y_0)=0+(0.1)f(0,0)=0.1$

$\therefore y(0.1)=0.1$

(ii) Modified Euler's method: $y_{n+1}=y_n+h[f(x_n+\frac{h}{2}, y_n+\frac{h}{2}f(x_n,y_n))]$

$y_2=y_1+h[f(x_1+\frac{h}{2}, y_1+\frac{h}{2}f(x_1,y_1))]$

$=0.1+(0.1)f(0.15,0.145)=0.1855$ $\therefore y(0.2)=0.1855$

$y_3=y_2+h[f(x_2+\frac{h}{2}, y_2+\frac{h}{2}f(x_2,y_2))]$

$=0.1855+(0.1)f(0.25,0.2262)=0.2629$

$\therefore y(0.3)=0.2629$

(iii) Milne's predictor formula: $y_{n+1,p}=y_{n-3}+\frac{4h}{3}[2y'_{n-2}-y'_{n-1}+2y'_n]$

$y_{4,p}=y_0+\frac{4h}{3}[2y_1'-y_2'+2y_3']$

$y_1'=1-y_1=0.9$, $y_2'=1-y_2=0.8145$, $y_3'=1-y_3=0.7371$

$\therefore y_{4,p}=\frac{4(0.1)}{3}[2(0.9)-0.8145+2(0.7371)]=0.328$

Milne's corrector formula: $y_{n+1,c}=y_{n-1}+\frac{h}{3}[y'_{n-1}+4y'_n+y'_{n+1}]$

$y_{4,c}=y_2+\frac{h}{3}[y_2'+4y_3'+y_4']$

$y_4'=1-y_4=0.672$

$\therefore y_{4,c}=0.1855+\frac{0.1}{3}[0.8145+4(0.7371)+0.672]=0.3333$

$\therefore y(0.4)=0.3333$

Adam's Predictor & Corrector methods:

$$y_{n+1,p}=y_n+\frac{h}{24}[55y'_n-59y'_{n-1}+37y'_{n-2}-9y'_{n-3}]$$

$$y_{n+1,c}=y_n+\frac{h}{24}[9y'_{n+1}+19y'_n-5y'_{n-1}+y'_{n-2}]$$

Problems:

① Given $\frac{dy}{dx}=x^2(1+y)$, $y(1)=1$, $y(1.1)=1.233$, $y(1.2)=1.548$, $y(1.3)=1.979$, evaluate $y(1.4)$ by Adam's method.

Sol: Given $x_0=1$, $x_1=1.1$, $x_2=1.2$, $x_3=1.3$, $h=0.1$

$y_0=1$, $y_1=1.233$, $y_2=1.548$, $y_3=1.979$

& $y'=x^2(1+y)$

Adam's predictor formula:

$$y_{n+1,p} = y_n + \frac{h}{24}\left[55y_n' - 59y_{n-1}' + 37y_{n-2}' - 9y_{n-3}'\right]$$

$$y_{4,p} = y_3 + \frac{h}{24}\left[55y_3' - 59y_2' + 37y_1' - 9y_0'\right]$$

$y_0' = x_0^2(1+y_0) = 2$, $y_1' = x_1^2(1+y_1) = 2.7019$, $y_2' = x_2^2(1+y_2) = 3.6691$

$y_3' = x_3^2(1+y_3) = 5.0345$

$$\therefore y_{4,p} = 1.979 + \frac{0.1}{24}\left[55(5.0345) - 59(3.6691) + 37(2.7019) - 9(2)\right] = 2.5723$$

Adam's corrector formula:

$$y_{n+1,c} = y_n + \frac{h}{24}\left[9y_{n+1}' + 19y_n' - 5y_{n-1}' + y_{n-2}'\right]$$

$$y_{4,c} = y_3 + \frac{h}{24}\left[9y_4' + 19y_3' - 5y_2' + y_1'\right]$$

$y_4' = x_4^2(1+y_4) = 7.0017$

$$\therefore y_{4,c} = 1.979 + \frac{0.1}{24}\left[9(7.0017) + 19(5.0345) - 5(3.6691) + 2.7019\right] = 2.5749$$

$\therefore y(1.4) = 2.5749$

---

② Find $y(0.1), y(0.2), y(0.3)$ from $\frac{dy}{dx} = xy + y^2$, $y(0) = 1$ by using R-K method & hence obtain $y(0.4)$ using Adam's method.

Sol: Given $f(x,y) = y' = xy + y^2$, $x_0 = 0, y_0 = 1, h = 0.1, x_1 = 0.1, x_2 = 0.2, x_3 = 0.3,$

$x_4 = 0.4$

R-K method:

First interval:

$k_1 = hf(x_0, y_0) = (0.1)f(0,1) = 0.1$

$k_2 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right] = (0.1)f(0.05, 1.05) = 0.1155$

$k_3 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right] = (0.1)f(0.05, 1.0578) = 0.1172$

$k_4 = hf\left[x_0 + h, y_0 + k_3\right] = (0.1)f(0.1, 1.1172) = 0.136$

$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.1169$

$\therefore y_1 = y_0 + \Delta y = 1.1169$ $\quad\therefore y(0.1) = 1.1169$

Second interval:

$k_1 = hf(x_1, y_1) = (0.1)f(0.1, 1.1169) = 0.1359$

$k_2 = hf\left[x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right] = (0.1)f(0.15, 1.1849) = 0.1582$

$k_3 = hf\left[x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right] = (0.1)f(0.15, 1.196) = 0.161$

$k_4 = hf[x_1+h,\ y_1+k_3] = (0.1)f(0.2, 1.2779) = 0.1889$

$\Delta y = \frac{1}{6}(k_1+2k_2+2k_3+k_4) = 0.1605$

$y_2 = y_1 + \Delta y = 1.2774 \qquad \therefore y(0.2) = 1.2774$

Third interval:

$k_1 = hf(x_2, y_2) = (0.1)f(0.2, 1.2774) = 0.1887$

$k_2 = hf[x_2+\frac{h}{2},\ y_2+\frac{k_1}{2}] = (0.1)f(0.25, 1.3718) = 0.2225$

$k_3 = hf[x_2+\frac{h}{2},\ y_2+\frac{k_2}{2}] = (0.1)f(0.25, 1.3887) = 0.2276$

$k_4 = hf[x_2+h,\ y_2+k_3] = (0.1)f(0.3, 1.505) = 0.2717$

$\Delta y = \frac{1}{6}(k_1+2k_2+2k_3+k_4) = 0.2268$

$y_3 = y_2 + \Delta y = 1.5042 \qquad \therefore y(0.3) = 1.5042$

Adam's Predictor formula: $y_{n+1,p} = y_n + \frac{h}{24}[55y'_n - 59y'_{n-1} + 37y'_{n-2} - 9y'_{n-3}]$

$y_{4,p} = y_3 + \frac{h}{24}[55y'_3 - 59y'_2 + 37y'_1 - 9y'_0]$

$y'_0 = x_0y_0 + y_0^2 = 1\ ,\quad y'_1 = x_1y_1 + y_1^2 = 1.3592\ ,\quad y'_2 = x_2y_2 + y_2^2 = 1.8872\ ,$

$y'_3 = x_3y_3 + y_3^2 = 2.7139$

$\therefore y_{4,p} = 1.5042 + \frac{0.1}{24}[55(2.7139) - 59(1.8872) + 37(1.3592) - 9(1)] = 1.8342$

Adam's corrector formula: $y_{n+1,c} = y_n + \frac{h}{24}[9y'_{n+1} + 19y'_n - 5y'_{n-1} + y'_{n-2}]$

$y_{4,c} = y_3 + \frac{h}{24}[9y'_4 + 19y'_3 - 5y'_2 + y'_1]$

$y'_4 = x_4y_4 + y_4^2 = 4.098$

$\therefore y_{4,c} = 1.5042 + \frac{0.1}{24}[9(4.098) + 19(2.7139) - 5(1.8872) + 1.3592)] = 1.8391$

$\therefore y(0.4) = 1.8391$

BOUNDARY VALUE PROBLEMS IN ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS

Defn.:

When the differential eqn. is to be solved satisfying the conditions specified at the end pts. of an interval, the problem is called boundary value problem.

Problems:

① Using the finite difference method, find $y(0.25)$, $y(0.5)$ & $y(0.75)$ satisfying the differential eqn. $\frac{d^2y}{dx^2}+y=x$, subject to the boundary conditions $y(0)=0$, $y(1)=2$.

Sol:

The given differential eqn. can be written as

$$y_i''+y_i=x_i \text{ ——— } ①$$

Central difference approximation: $y_i''=\frac{y_{i-1}-2y_i+y_{i+1}}{h^2}$

$$\therefore ① \Rightarrow \frac{y_{i-1}-2y_i+y_{i+1}}{h^2}+y_i=x_i$$

$$\Rightarrow y_{i-1}-(2-h^2)y_i+y_{i+1}=h^2x_i$$

$$\Rightarrow y_{i-1}-1.9375y_i+y_{i+1}=0.0625x_i \text{ ——— } ②$$

Given $x_0=0$, $y_0=0$, $x_1=0.25$, $x_2=0.5$, $x_3=0.75$, $x_4=1$, $y_4=2$

Put $i=1$ in ②, $y_0-1.9375y_1+y_2=0.0625x_1$

$$\Rightarrow -1.9375y_1+y_2=0.0156 \text{ ——— } ③$$

Put $i=2$ in ②, $y_1-1.9375y_2+y_3=0.0625x_2$

$$\Rightarrow y_1-1.9375y_2+y_3=0.0313 \text{ ——— } ④$$

Put $i=3$ in ②, $y_2-1.9375y_3+y_4=0.0625x_3$

$$\Rightarrow y_2-1.9375y_3=-1.9531 \text{ ——— } ⑤$$

$④\times 1.9375 \Rightarrow 1.9375y_1-3.7539y_2+1.9375y_3=0.0606$

$-1.9375y_1+y_2 \qquad = 0.0156$ ——— ③

$$-2.7539y_2+1.9375y_3=0.0762 \text{ ——— } ⑥$$

$$⑤+⑥ \Rightarrow -1.7539y_2=-1.8769 \Rightarrow \boxed{y_2=1.0701}$$

Subs. y_2 in ③, $-1.9375y_1 + 1.0701 = 0.0156$

$$\Rightarrow \boxed{y_1 = 0.5443}$$

Subs. y_2 in ⑤, $1.0701 - 1.9375y_3 = -1.9531$

$$\Rightarrow \boxed{y_3 = 1.5604}$$

Hence $y(0.25) = 0.5443$, $y(0.5) = 1.0701$, $y(0.75) = 1.5604$

② Solve the eqn. $y''(x) - xy(x) = 0$ for $y(x_i)$, $x_i = 0, \frac{1}{3}, \frac{2}{3}$, given that $y(0) + y'(0) = 1$ & $y(1) = 1$.

Sol:

The given eqn. can be rewritten as $y_i'' - x_i y_i = 0$ —①

Central difference approximation: $y_i'' = \dfrac{y_{i-1} - 2y_i + y_{i+1}}{h^2}$, $y_i' = \dfrac{y_{i+1} - y_{i-1}}{2h}$

$$\therefore ① \Rightarrow \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} = x_i y_i$$

$$\Rightarrow y_{i-1} - (2 + x_i h^2)y_i + y_{i+1} = 0$$

$$\Rightarrow y_{i-1} - \left(2 + \frac{1}{9}x_i\right)y_i + y_{i+1} = 0 \quad \text{—②}$$

Given $y_0 + y_0' = 1 \Rightarrow y_0 + \dfrac{y_1 - y_{-1}}{2/3} = 1 \Rightarrow 3y_{-1} = 2y_0 + 3y_1 - 2$

$$\Rightarrow y_{-1} = \frac{1}{3}(2y_0 + 3y_1 - 2)$$

Given $x_0 = 0$, $x_1 = 1/3$, $x_2 = 2/3$, $x_3 = 1$, $y_3 = 1$, $h = 1/3$

Put $i = 0$ in ②, $y_{-1} - 2y_0 + y_1 = 0 \Rightarrow \dfrac{2y_0 + 3y_1 - 2}{3} - 2y_0 + y_1 = 0$

$$\Rightarrow 3y_1 - 2y_0 - 1 = 0 \quad \text{—③}$$

Put $i = 1$ in ②, $y_0 - \frac{55}{27}y_1 + y_2 = 0$ —④

Put $i = 2$ in ②, $y_1 - \frac{56}{27}y_2 + 1 = 0$ —⑤

$$④ \times 2 \Rightarrow \begin{array}{r} 2y_0 - 4.0741y_1 + 2y_2 = 0 \\ -2y_0 + 3y_1 - 1 = 0 \quad \text{—③} \\ \hline -1.0741y_1 + 2y_2 = 1 \end{array}$$

$$⑤ \times 1.0741 \Rightarrow \begin{array}{r} 1.0741y_1 - 2.2278y_2 = -1.0741 \\ \hline -0.2278y_2 = -0.0741 \end{array}$$

$$\boxed{y_2 = 0.3253}$$

Subs. y_2 in ⑤, $y_1 = 0.6747 - 1 = -0.3253 \Rightarrow \boxed{y_1 = -0.3253}$

Subs/. y_1 in ③, $2y_0=-0.9759-1$ $\Rightarrow$ $\boxed{y_0=-0.988}$

Hence $y(0)=-0.988$, $y(1/3)=-0.3253$, $y(2/3)=0.3253$

③ Solve $y''-xy=0$ given $y(0)=-1$, $y(1)=2$ by finite difference method taking $n=2$.

Sol: If $n=2$, then $h=1/2$ since range is $(0,1)$.

Here $x_0=0$, $x_1=0.5$, $x_2=1$, $y_0=-1$, $y_2=2$

Central difference approximation: $y_i''=\frac{y_{i-1}-2y_i+y_{i+1}}{h^2}$

The given eqn/. can be rewritten as $y_i''-x_iy_i=0$

$$\frac{y_{i-1}-2y_i+y_{i+1}}{h^2}=x_iy_i \Rightarrow y_{i+1}-(2+0.25x_i)y_i+y_{i-1}=0 \quad \text{—①}$$

Put $i=1$ in ①, $y_2-(2+0.25x_1)y_1+y_0=0$

$\Rightarrow 1-2.125y_1=0 \Rightarrow \boxed{y_1=0.4706}$

$\therefore y(0.5)=0.4706$

Classification of PDE of second order:

The general linear PDE of second order can be written as

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=0$$

(ii) $Au_{xx}+Bu_{xy}+Cu_{yy}+Du_x+Eu_y+Fu=0$ — ① where A,B,C,D,E,F are the funs/. of x & y.

Eqn/. ① is said to be (i) elliptic if $B^2-4AC<0$

(ii) parabolic if $B^2-4AC=0$

(iii) hyperbolic if $B^2-4AC>0$

Examples:

Elliptic type: $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ (Laplace eqn/. in two dimension)

$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f(x,y)$ (Poisson eqn/.)

Parabolic type: $\frac{\partial^2 u}{\partial x^2}=\frac{1}{\alpha^2}\frac{\partial u}{\partial t}$ (One dimensional heat eqn/.)

Hyperbolic type: $\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial^2 u}{\partial t^2}$ (One dimensional wave eqn/.)

Problem:

Classify the following: (i) $u_{xx} + 4u_{xy} + (x^2+4y^2)u_{yy} = \sin(x+y)$

(ii) $(x+1)u_{xx} - 2(x+2)u_{xy} + (x+3)u_{yy} = 0$

Sol: (i) Given $A=1$, $B=4$, $C=x^2+4y^2$

$B^2-4AC = 4[4-x^2-4y^2]$

The eqn/. is elliptic if $4-x^2-4y^2<0$ (ie) $x^2+4y^2>4$

(ie) $\frac{x^2}{4}+\frac{y^2}{1}>1$

$\therefore$ It is elliptic outside the ellipse $\frac{x^2}{4}+\frac{y^2}{1}=1$.

It is hyperbolic inside the ellipse $\frac{x^2}{4}+\frac{y^2}{1}=1$.

It is parabolic on the ellipse $\frac{x^2}{4}+\frac{y^2}{1}=1$

(ii) Given $A=x+1$, $B=-2(x+2)$, $C=x+3$

$B^2-4AC = 4(x+2)^2-4(x+1)(x+3) = 4 > 0$

$\therefore$ The eqn/. is hyperbolic at all pts. of the region.

Bender-Schmidt's Difference eqn/.:

One dimensional heat eqn/.: (Explicit method)

$$\frac{\partial^2 u}{\partial x^2} - a\frac{\partial u}{\partial t} = 0$$

$$u_{xx} = \frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{h^2}$$

$$u_t = \frac{u_{i,j+1}-u_{i,j}}{k}$$

$u_{i,j+1} = \lambda u_{i+1,j} + (1-2\lambda)u_{i,j} + \lambda u_{i-1,j}$ ① where $\lambda = \frac{k}{ah^2}$

This eqn/. is called explicit formula. It is valid if $0<\lambda\le 1/2$.

Take $\lambda = 1/2$,

$u_{i,j+1} = \frac{1}{2}[u_{i+1,j}+u_{i-1,j}]$ ② where $\lambda = \frac{1}{2} = \frac{k}{ah^2} \Rightarrow k = \frac{a}{2}h^2$

Eqn/. ② is called Bender-Schmidt recurrence eqn/.

Problems:

① Solve $\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial t} = 0$ given $u(0,t)=0$, $u(4,t)=0$, $u(x,0)=x(4-x)$.
Assume $h=1$. Find the values of u upto $t=5$.

Sol:

Given $u_{xx} = 2u_t$, $a=2$, $h=1$, $\lambda = \frac{1}{2}$

$\therefore k = \frac{ah^2}{2} = 1$

Given $u(0,t)=0$, $u(4,t)=0$, $u(x,0)=x(4-x)$

Bender-Schmidt recurrence eqn. is $u_{i,j+1} = \frac{1}{2}\left[u_{i-1,j} + u_{i+1,j}\right]$

The values of $u_{i,j}$ are tabulated below:

x-direction

t-direction

j\i	0	1	2	3	4
0	0	3	4	3	0
1	0	2	3	2	0
2	0	1.5	2	1.5	0
3	0	1	1.5	1	0
4	0	0.75	1	0.75	0
5	0	0.5	0.75	0.5	0

---

② Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $0 \le x \le 1$, $t \ge 0$ with $u(x,0) = x(1-x)$, $0 < x < 1$ & $u(0,t) = u(1,t) = 0$ $\forall$ t using explicit method with $\Delta x = 0.2$ for 3 time steps.

Sol:

Given $u_{xx} = u_t$, $a=1$, $h=0.2$, $\lambda = \frac{1}{2}$

$k = \frac{ah^2}{2} = 0.02$

Bender-Schmidt recurrence eqn. is $u_{i,j+1} = \frac{1}{2}\left[u_{i-1,j} + u_{i+1,j}\right]$

Given $u(x,0) = x(1-x)$, $0 < x < 1$, $u(0,t) = u(1,t) = 0$ $\forall$ t

The values of $u_{i,j}$ are tabulated below:

x-direction

t-direction

j\i	0	0.2	0.4	0.6	0.8	1
0	0	0.16	0.24	0.24	0.16	0
0.02	0	0.12	0.2	0.2	0.12	0
0.04	0	0.1	0.16	0.16	0.1	0
0.06	0	0.08	0.13	0.13	0.08	0

---

③ Solve $\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}$, $0 \le x \le 12$, $0 \le t \le 12$, with boundary conditions & initial conditions $u(x,0) = \frac{1}{4}x(15-x)$, $0 \le x \le 12$, $u(0,t) = 0$, $u(12,t) = 9$, $0 \le t \le 12$. Using Schmidt relation using $h = k = 3$.

Sol:

Given $u_{xx} = 2u_t$, $a = 2$, $h = k = 3$

$$\lambda = \frac{k}{ah^2} = \frac{1}{6}$$

Explicit formula: $u_{i,j+1} = \lambda u_{i+1,j} + (1-2\lambda)u_{i,j} + \lambda u_{i-1,j}$

$$u_{i,j+1} = \frac{1}{6}\left[u_{i+1,j} + u_{i-1,j}\right] + \frac{2}{3}u_{i,j}$$

x-direction

t-direction

j\i	0	3	6	9	12
0	0	9	13.5	13.5	9
3	0	8.25	12.75	12.75	9
6	0	7.625	12	12.125	9
9	0	7.0833	11.2917	11.5833	9
12	0	6.6042	10.6389	11.1042	9

Crank-Nicolson's difference eqn.:

One dimensional heat eqn.: [Implicit method]

$$u_{xx} = au_t \quad \text{—①}$$

At $u_{i,j}$, $u_{xx} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$

At $u_{i,j+1}$, $u_{xx} = \frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{h^2}$

Taking the average of these 2 values,

$$u_{xx} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j} + u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{2h^2} \quad \& \quad u_t = \frac{u_{i,j+1} - u_{i,j}}{k}$$

$$\therefore ① \Rightarrow \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j} + u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{2h^2} = a\,\frac{u_{i,j+1} - u_{i,j}}{k}$$

Setting $\frac{k}{ah^2} = \lambda$, the above eqn/. reduces to

$$2(\lambda+1)u_{i,j+1} = \lambda\left[u_{i+1,j+1} + u_{i-1,j+1} + u_{i+1,j} + u_{i-1,j}\right] - 2(\lambda-1)u_{i,j}$$

This eqn/. is known as Crank-Nicolson's difference eqn/. (or) Implicit formula.

Take $\lambda = 1$ (ie) $k = ah^2$

The Crank-Nicolson's formula reduces to

$$u_{i,j+1} = \frac{1}{4}\left[u_{i+1,j+1} + u_{i-1,j+1} + u_{i+1,j} + u_{i-1,j}\right]$$

Problems:

① Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ in $0 \le x \le 5$, $t \ge 0$ given that $u(x,0) = 20$, $u(0,t) = 0$, $u(5,t) = 100$. Compute u for the time-step with $h = 1$ by Crank-Nicolson method.

Sol: Given $a = 1$, $h = 1$, $\lambda = \frac{k}{ah^2}$

Put $\lambda = 1$, $k = ah^2$

The Crank-Nicolson formula reduces to

$$u_{i,j+1} = \frac{1}{4}\left[u_{i+1,j+1} + u_{i-1,j+1} + u_{i+1,j} + u_{i-1,j}\right]$$

j\i	0	1	2	3	4	5
0	0	20	20	20	20	100
1	0	u_1	u_2	u_3	u_4	100

$u_1 = \frac{1}{4}[0 + 20 + 0 + u_2] \Rightarrow 4u_1 - u_2 = 20$ — ①

$u_2 = \frac{1}{4}[20 + 20 + u_1 + u_3] \Rightarrow u_1 - 4u_2 + u_3 = -40$ — ②

$u_3 = \frac{1}{4}[20 + 20 + u_2 + u_4] \Rightarrow u_2 - 4u_3 + u_4 = -40$ — ③

$u_4 = \frac{1}{4}[20 + 100 + u_3 + 100] \Rightarrow u_3 - 4u_4 = -220$ — ④

$$② \times 4 \Rightarrow 4u_1 - 16u_2 + 4u_3 = -160$$

$$4u_1 - u_2 \qquad = 20 \text{ — } ①$$

$$(-) \quad (+) \qquad (-)$$

$$-15u_2 + 4u_3 = -180$$

$$③ \times 15 \Rightarrow 15u_2 - 60u_3 + 15u_4 = -600$$

$$-56u_3 + 15u_4 = -780$$

$$④ \times 56 \Rightarrow 56u_3 - 224u_4 = -12320$$

$$-209u_4 = -13100 \Rightarrow \boxed{u_4 = 62.6794}$$

Subs. u_4 in ④, $u_3 = -220 + (4 \times 62.6794)$

$$\boxed{u_3 = 30.7176}$$

Subs. u_3 & u_4 in ③, $u_2 = -40 + 4(30.7176) - 62.6794$

$$\boxed{u_2 = 20.191}$$

Subs. u_2 in ①, $4u_1 = 20 + 20.191 \Rightarrow \boxed{u_1 = 10.0478}$

Hence $u_1 = 10.0478$, $u_2 = 20.191$, $u_3 = 30.7176$, $u_4 = 62.6794$

② Using Crank-Nicolson method, solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, subject to $u(x,0) = 0$, $u(0,t) = 0$ & $u(1,t) = t$, (i) taking $h = 0.5$ & $k = 1/8$ & (ii) $h = 1/4$ & $k = 1/8$.

Sol:

(i) Given $a = 1$, $h = 0.5$, $k = 1/8$

$$\lambda = \frac{k}{ah^2} = 1/2$$

Crank-Nicolson's difference eqn. is

$$2(\lambda+1)u_{i,j+1} = \lambda[u_{i+1,j+1} + u_{i-1,j+1} + u_{i+1,j} + u_{i-1,j}] - 2(\lambda-1)u_{i,j}$$

$$u_{i,j+1} = \frac{1}{3}\left[\frac{1}{2}(u_{i+1,j+1} + u_{i-1,j+1} + u_{i+1,j} + u_{i-1,j}) + u_{i,j}\right]$$

$j \backslash i$	0	0.5	1
0	0	0	0
$1/8$	0	u_1	$1/8$

$$u_1 = \frac{1}{3}\left[\frac{1}{2}\left(\frac{1}{8} + 0 + 0 + 0\right) + 0\right] = \frac{1}{3} \times \frac{1}{16} = \frac{1}{48} = 0.0208$$

(ii) Given $a=1$, $h=\frac{1}{4}$, $k=\frac{1}{8}$

$$\lambda=\frac{k}{ah^2}=2$$

Crank-Nicolson's difference eqn. is

$$u_{i,j+1}=\frac{1}{3}\left[u_{i+1,j+1}+u_{i-1,j+1}+u_{i+1,j}+u_{i-1,j}-u_{i,j}\right]$$

j\i	0	0.25	0.5	0.75	1
0	0	0	0	0	0
1/8	0	u_1	u_2	u_3	1/8

$$u_1=\frac{1}{3}\left[u_2+0+0+0-0\right]=\frac{u_2}{3} \Rightarrow 3u_1-u_2=0 \text{ — ①}$$

$$u_2=\frac{1}{3}\left[0+0+u_1+u_3-0\right] \Rightarrow u_1-3u_2+u_3=0 \text{ — ②}$$

$$u_3=\frac{1}{3}\left[0+0+u_2+\frac{1}{8}-0\right] \Rightarrow 24u_3=8u_2+1 \Rightarrow 8u_2-24u_3+1=0 \text{ — ③}$$

② × 3 ⇒ $3u_1-9u_2+3u_3=0$

$3u_1-u_2 \quad = 0$ — ①

(−) (+) (−)

$-8u_2+3u_3=0$

$8u_2-24u_3=-1$ — ③

$$-21u_3=-1 \Rightarrow u_3=\frac{1}{21}=0.0476 \Rightarrow \boxed{u_3=0.0476}$$

Subs/. u_3 in ③, $8u_2=-1+24(0.0476) \Rightarrow \boxed{u_2=0.0178}$

Subs/. u_2 in ①, $3u_1=0.0178 \Rightarrow \boxed{u_1=0.0059}$

Hence $u_1=0.0059$, $u_2=0.0178$, $u_3=0.0476$

---

③ Solve $\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}$, $0\le x\le 2$, $t\ge 0$, $u(0,t)=u(2,t)=0$, $t\ge 0$ & $u(x,0)=\sin\frac{\pi x}{2}$ $0\le x\le 2$ using $\Delta x=0.5$ & $\Delta t=0.25$ for 2 time steps by Crank-Nicolson implicit finite difference method.

Sol:

Given $a=1$, $h=0.5$, $k=0.25$

$$\lambda=\frac{k}{ah^2}=1$$

Crank-Nicolson's formula is $u_{i,j+1}=\frac{1}{4}\left[u_{i-1,j+1}+u_{i+1,j+1}+u_{i-1,j}+u_{i+1,j}\right]$

$j \backslash i$	0	0.5	1	1.5	2
0	0	0.7071	1	0.7071	0
0.25	0	u_1	u_2	u_3	0
0.5	0	u_4	u_5	u_6	0

$u_1 = \frac{1}{4}[0+1+0+u_2] \Rightarrow 4u_1 - u_2 = 1$ ①

$u_2 = \frac{1}{4}[0.7071+0.7071+u_1+u_3] \Rightarrow u_1 - 4u_2 + u_3 = -1.4142$ ②

$u_3 = \frac{1}{4}[1+0+u_2+0] \Rightarrow u_2 - 4u_3 = -1$ ③

$u_4 = \frac{1}{4}[0+u_2+0+u_5] \Rightarrow u_2 - 4u_4 + u_5 = 0$ ④

$u_5 = \frac{1}{4}[u_1+u_3+u_4+u_6] \Rightarrow u_1+u_3+u_4-4u_5+u_6 = 0$ ⑤

$u_6 = \frac{1}{4}[u_2+0+u_5+0] \Rightarrow u_2+u_5-4u_6 = 0$ ⑥

② × 4 ⇒ $4u_1 - 16u_2 + 4u_3 = -5.6568$

$4u_1 - u_2 = 1$

$(-) \quad (+) \qquad (-)$

$-15u_2 + 4u_3 = -6.6568$

③ ×15 ⇒ $15u_2 - 60u_3 = -15$

$-56u_3 = -21.6568 \Rightarrow \boxed{u_3 = 0.3867}$

Subs. u_3 in ③, $\boxed{u_2 = 0.5468}$

Subs. u_2 in ①, $4u_1 = 1.5468 \Rightarrow \boxed{u_1 = 0.3867}$

Subs. u_1 & u_3 in ⑤, $u_4 - 4u_5 + u_6 = -0.7734$ ⑦

Subs. u_2 in ④, $-4u_4 + u_5 = -0.5468$ ⑧

Subs. u_2 in ⑥, $u_5 - 4u_6 = -0.5468$ ⑨

⑦ × 4 ⇒ $4u_4 - 16u_5 + 4u_6 = -3.0936$

$-4u_4 + u_5 = -0.5468$ ⑧

$-15u_5 + 4u_6 = -3.6404$

⑨ ×15 ⇒ $15u_5 - 60u_6 = -8.202$

$-56u_6 = -11.8424 \Rightarrow \boxed{u_6 = 0.2115}$

Subs. u_6 in ⑨, $\boxed{u_5 = 0.2992}$

Subs. u_5 in ⑧, $-4u_4 = -0.846 \Rightarrow \boxed{u_4 = 0.2115}$

Hence $u_1 = 0.3867$, $u_2 = 0.5468$, $u_3 = 0.3867$, $u_4 = 0.2115$, $u_5 = 0.2992$, $u_6 = 0.2115$

One dimensional wave eqn.:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \Rightarrow a^2 u_{xx} - u_{tt} = 0 \quad — ①$$

$$u_{xx} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} \quad , \quad u_{tt} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2}$$

$$\therefore ① \Rightarrow u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j-1} \quad \text{[Explicit formula]}$$

Problems:

① Solve $u_{tt} = u_{xx}$ upto $t = 0.5$ with a spacing of 0.1 subject to $u(0,t) = 0$, $u(1,t) = 0$, $u_t(x,0) = 0$ & $u(x,0) = 10 + x(1-x)$.

Sol: Given $a^2 = 1 \Rightarrow a = 1$, $h = 0.1$, $k = \frac{h}{a} = 0.1$

$$u_t(x,0) = 0 \Rightarrow u_{i,1} = \frac{u_{i+1,0} + u_{i-1,0}}{2}$$

Explicit formula is $u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j-1}$

j\i	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0	0	10.09	10.16	10.21	10.24	10.25	10.24	10.21	10.16	10.09	0
0.1	0	5.08	10.15	10.2	10.23	10.24	10.23	10.2	10.15	5.08	0
0.2	0	0.06	5.12	10.17	10.2	10.21	10.2	10.17	5.12	0.06	0
0.3	0	0.04	0.08	5.12	10.15	10.16	10.15	5.12	0.08	0.04	0
0.4	0	0.02	0.04	0.06	5.08	10.09	5.08	0.06	0.04	0.02	0
0.5	0	0	0	0	0	0	0	0	0	0	0

② Solve $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$, $0 < x < 1$, $t \geq 0$, $u(0,t) = 0$, $u(1,t) = 0$, $u(x,0) = x - x^2$, $\frac{\partial u}{\partial t}(x,0) = 0$ taking $h = 0.2$ upto one half of the period of vibration by taking appropriate time step.

Sol: Given $a^2 = 1 \Rightarrow a = 1$, $h = 0.2$, $k = \frac{h}{a} = 0.2$

$$u_t(x,0) = 0 \Rightarrow u_{i,1} = \frac{u_{i+1,0} + u_{i-1,0}}{2}$$

Explicit formula is $u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j-1}$

Period of oscillation $= \frac{2l}{a} = 2$ secs

$\therefore$ One half of period $= \frac{2}{2} = 1$ sec

j\i	0	0.2	0.4	0.6	0.8	1
0	0	0.16	0.24	0.24	0.16	0
0.2	0	0.12	0.2	0.2	0.12	0
0.4	0	0.04	0.08	0.08	0.04	0
0.6	0	−0.04	−0.08	−0.08	−0.04	0
0.8	0	−0.12	−0.2	−0.2	−0.12	0
1	0	−0.16	−0.24	−0.24	−0.16	0

③ Solve $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$, $0<x<1$, $t>0$, given $u(x,0)=100(x-x^2)$, $\frac{\partial u}{\partial t}(x,0)=0$, $u(0,t)=u(1,t)=0, t>0$ by finite difference method for once time step with $h=0.25$.

Sol:

Given $a^2=1 \Rightarrow a=1$, $h=0.25$, $k=\frac{h}{a}=0.25$

$$u_t(x,0)=0 \Rightarrow u_{i,1} = \frac{u_{i+1,0}+u_{i-1,0}}{2}$$

Explicit formula is $u_{i,j+1} = u_{i+1,j}+u_{i-1,j}-u_{i,j-1}$

j\i	0	0.25	0.5	0.75	1
0	0	18.75	25	18.75	0
0.25	0	12.5	18.75	12.5	0
0.5	0	0	0	0	0

Elliptic eqns:

Laplace eqn. is $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ (ie) $u_{xx}+u_{yy}=0$ —①

WKT $u_{xx} = \frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{h^2}$, $u_{yy} = \frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{k^2}$

$$\therefore ① \Rightarrow \frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{h^2} + \frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{k^2} = 0$$

Take $k=h$

$$u_{i,j} = \frac{1}{4}\left[u_{i+1,j}+u_{i-1,j}+u_{i,j+1}+u_{i,j-1}\right]$$

This is called standard five point formula. [SFPF]

Diagonal five point formula: [DFPF]

$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j-1} + u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1}\right]$$

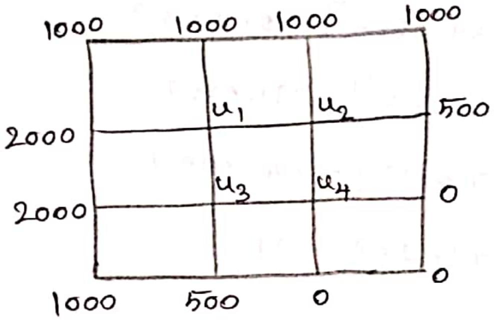
Liebmann's iteration process:

$$u_{i,j}^{(n+1)} = \frac{1}{4}\left[u_{i+1,j}^{(n)} + u_{i-1,j}^{(n+1)} + u_{i,j-1}^{(n)} + u_{i,j+1}^{(n+1)}\right]$$

where the superscript of n denotes the iteration no/..

Problems:

① Obtain a finite difference scheme to solve the Laplace eqn/. $\nabla^2 u=0$ at the pivotal pts in the square shown fitted with square mesh. Use Leibmann's iteration procedure. (5 iterations only)



Sol: Take $u_4=0$

Rough values:

$u_1=\frac{1}{4}[1000+1000+2000+u_4]=1000$ (DFPF)

$u_2=\frac{1}{4}[1000+500+u_1+u_4]=625$ (SFPF)

$u_3=\frac{1}{4}[2500+u_1+u_4]=875$ (SFPF)

$u_4=\frac{1}{4}[0+u_2+u_3]=375$ (SFPF)

Here after we apply only SFPF.

1st iteration:

$u_1^{(1)}=\frac{1}{4}[1000+2000+u_2+u_3]=1125$

$u_2^{(1)}=\frac{1}{4}[1500+1125+375]=750$

$u_3^{(1)}=\frac{1}{4}[2500+1125+375]=1000$

$u_4^{(1)}=\frac{1}{4}[750+1000]=437.5$

2nd iteration:

$u_1^{(2)}=\frac{1}{4}[3000+750+1000]=1187.5$

$u_2^{(2)} = \frac{1}{4}[1500 + 1187.5 + 437.5] = 781.25$

$u_3^{(2)} = \frac{1}{4}[2500 + 1187.5 + 437.5] = 1031.25$

$u_4^{(2)} = \frac{1}{4}[781.25 + 1031.25] = 453.125$

3rd iteration:

$u_1^{(3)} = \frac{1}{4}[3000 + 781.25 + 1031.25] = 1203.125$

$u_2^{(3)} = \frac{1}{4}[1500 + 1203.125 + 453.125] = 789.0625$

$u_3^{(3)} = \frac{1}{4}[2500 + 1203.125 + 453.125] = 1039.0625$

$u_4^{(3)} = \frac{1}{4}[789.0625 + 1039.0625] = 457.0313$

4th iteration:

$u_1^{(4)} = \frac{1}{4}[3000 + 789.0625 + 1039.0625] = 1207.0313$

$u_2^{(4)} = \frac{1}{4}[1500 + 1207.0313 + 457.0313] = 791.0157$

$u_3^{(4)} = \frac{1}{4}[2500 + 1207.0313 + 457.0313] = 1041.0157$

$u_4^{(4)} = \frac{1}{4}[791.0157 + 1041.0157] = 458.0079$

5th iteration:

$a_1^{(5)} = \frac{1}{4}[3000 + 791.0157 + 1041.0157] = 1208.0079$

$u_2^{(5)} = \frac{1}{4}[1500 + 1208.0079 + 458.0079] = 791.504$

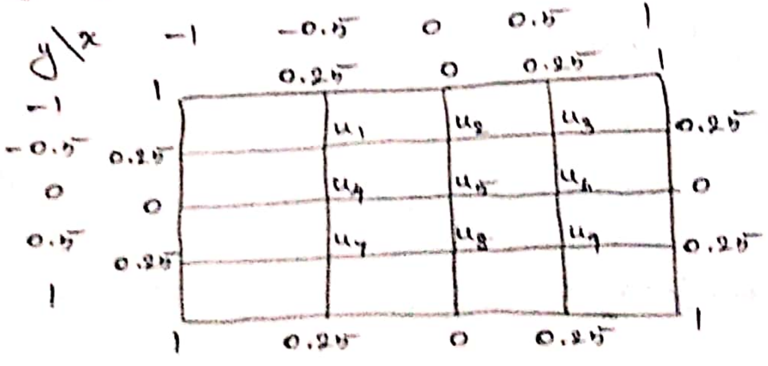
$u_3^{(5)} = \frac{1}{4}[2500 + 1208.0079 + 458.0079] = 1041.504$

$u_4^{(5)} = \frac{1}{4}[791.504 + 1041.504] = 458.252$

Hence $u_1 = 1208.0079$, $u_2 = 791.504$, $u_3 = 1041.504$, $u_4 = 458.252$

② Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ in $|x| \le 1$, $|y| \le 1$ with $h = 1/2$ & $u(x, \pm 1) = x^2$, $u(\pm 1, y) = y^2$. (ie) (i) $u(x,1) = x^2, -1 \le x \le 1$ (ii) $u(x,-1) = x^2, -1 \le x \le 1$ (iii) $u(1,y) = y^2, -1 \le y \le 1$ (iv) $u(-1,y) = y^2, -1 \le y \le 1$.

Sol:



Rough values:

$u_5 = \frac{1}{4}[0+0+0+0] = 0$ [SFPF]

$u_1 = \frac{1}{4}[1+0+0+u_5] = 0.25$ [DFPF]

$u_3 = \frac{1}{4}[0+1+u_5+0] = 0.25$ [DFPF]

$u_7 = \frac{1}{4}[0+u_5+1+0] = 0.25$ [DFPF]

$u_9 = \frac{1}{4}[u_5+0+0+1] = 0.25$ [DFPF]

$u_2 = \frac{1}{4}[0+u_1+u_3+u_5] = 0.125$ [SFPF]

$u_4 = \frac{1}{4}[0+u_1+u_5+u_7] = 0.125$ [SFPF]

$u_6 = \frac{1}{4}[0+u_3+u_5+u_9] = 0.125$ [SFPF]

$u_8 = \frac{1}{4}[0+u_5+u_7+u_9] = 0.125$ [SFPF]

In all further calculations we use SFPF & the latest available values.

1st iteration:

$u_1^{(1)} = \frac{1}{4}[0.25+0.25+u_2+u_4] = 0.1875$

$u_2^{(1)} = \frac{1}{4}[0+u_1+u_3+u_5] = 0.1094$

$u_3^{(1)} = \frac{1}{4}[0.5+u_2+u_6] = 0.1836$

$u_4^{(1)} = \frac{1}{4}[0+u_1+u_5+u_7] = 0.1094$

$u_5^{(1)} = \frac{1}{4}[u_2+u_4+u_6+u_8] = 0.1172$

$u_6^{(1)} = \frac{1}{4}[0+u_3+u_5+u_9] = 0.1377$

$u_7^{(1)} = \frac{1}{4}[0.5+u_4+u_8] = 0.1836$

$u_8^{(1)} = \frac{1}{4}[0+u_5+u_7+u_9] = 0.1377$

$u_9^{(1)} = \frac{1}{4}[0.5+u_6+u_8] = 0.1939$

2nd iteration:

$u_1^{(2)} = \frac{1}{4}[0.5+u_2+u_4] = 0.1797$

$u_2^{(2)} = \frac{1}{4}[u_1+u_3+u_5] = 0.1201$

$u_3^{(2)} = \frac{1}{4}[0.5+u_2+u_6] = 0.1895$

$u_4^{(2)} = \frac{1}{4}[u_1+u_5+u_7] = 0.1201$

$u_5^{(2)} = \frac{1}{4}[u_2+u_4+u_6+u_8] = 0.1289$

$u_6^{(2)} = \frac{1}{4}[u_3+u_5+u_9] = 0.1281$

$u_7^{(2)} = \frac{1}{4}[0.5+u_4+u_8] = 0.1895$

$u_8^{(2)} = \frac{1}{4}[u_5 + u_7 + u_9] = 0.1281$

$u_9^{(2)} = \frac{1}{4}[0.5 + u_6 + u_8] = 0.1891$

3rd iteration:

$u_1^{(3)} = 0.1851$

$u_2^{(3)} = 0.1259$

$u_3^{(3)} = 0.1885$

$u_4^{(3)} = 0.1259$

$u_5^{(3)} = 0.127$

$u_6^{(3)} = 0.1262$

$u_7^{(3)} = 0.1885$

$u_8^{(3)} = 0.1262$

$u_9^{(3)} = 0.1881$

4th iteration:

$u_1^{(4)} = 0.188$

$u_2^{(4)} = 0.1259$

$u_3^{(4)} = 0.188$

$u_4^{(4)} = 0.1259$

$u_5^{(4)} = 0.1261$

$u_6^{(4)} = 0.1256$

$u_7^{(4)} = 0.188$

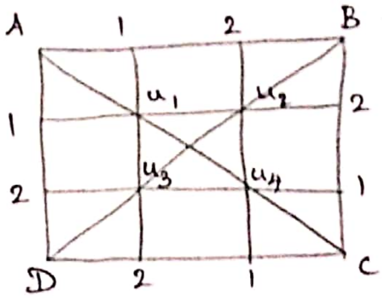
$u_8^{(4)} = 0.1256$

$u_9^{(4)} = 0.1878$

Hence $u_1 = 0.19$, $u_2 = 0.13$, $u_3 = 0.19$, $u_4 = 0.13$, $u_5 = 0.13$, $u_6 = 0.13$, $u_7 = 0.19$, $u_8 = 0.13$, $u_9 = 0.19$ (correct to 2 decimal places)

---

③ Solve $u_{xx} + u_{yy} = 0$ for the following square mesh with boundary conditions as shown below. Iterate until the maximum difference between successive values at any grid pt. is less than 0.001.



Sol: From the above figure, we see that it is symmetrical about the diagonals AC & BD. By symmetry $u_1 = u_4$, $u_2 = u_3$.

∴ We have to find only 2 values u_1 & u_2. Assume $u_2 = 0$.

Rough values:

$u_1 = \frac{1}{4}[1 + 1 + u_2 + u_2] = 0.5$ [SFPF]

$u_2 = \frac{1}{4}[2 + 2 + u_1 + u_1] = 1.25$ [SFPF]

1st iteration:

$u_1^{(1)} = \frac{1}{4}[2 + 1.25 + 1.25] = 1.125$; $u_2^{(1)} = \frac{1}{4}[4 + 1.125 + 1.125] = 1.5625$

2nd iteration:

$u_1^{(2)} = \frac{1}{4}[2+u_2+u_2] = 1.2813$; $u_2^{(2)} = \frac{1}{4}[4+u_1+u_1] = 1.6407$

3rd iteration:

$u_1^{(3)} = \frac{1}{4}[2+u_2+u_2] = 1.3204$; $u_2^{(3)} = 1.6602$

4th iteration:

$u_1^{(4)} = 1.3301$; $u_2^{(4)} = 1.6651$

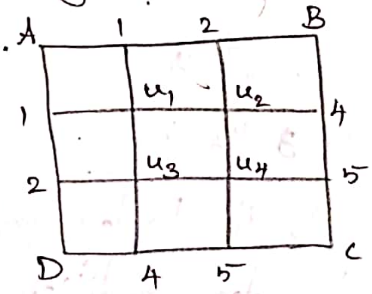
5th iteration: $u_1^{(5)} = 1.3326$; $u_2^{(5)} = 1.6663$

6th iteration: $u_1^{(6)} = 1.3332$; $u_2^{(6)} = 1.6666$

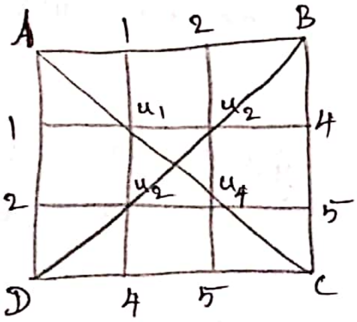
The difference between 2 consecutive values of u_1 is 0.0006 & that between 2 consecutive values of u_2 is 0.0003 which are less than 0.001..

Hence $u_1 = 1.3332$, $u_2 = 1.6666$

④ Solve $u_{xx} + u_{yy} = 0$ for the following square mesh with boundary values as shown in the figure below.



Sol:



The boundary values are symmetrical about the diagonal AC but not about BD. By symmetry $u_2 = u_3$, $u_1 \neq u_4$. ∴ We need to find u_1, u_2, u_4 only.

Assume $u_2 = 0$.

Rough values:

$u_1 = \frac{1}{4}[1+1+u_2+u_2] = 0.5$ [SFPF]

$u_2 = 0$; $u_4 = \frac{1}{4}[10+2u_2] = 2.5$

1st iteration:

$u_1^{(1)} = \frac{1}{4}[2+2u_2] = 0.5$; $u_2^{(1)} = \frac{1}{4}[6+u_1+u_4] = 2.25$; $u_4^{(1)} = \frac{1}{4}[10+2u_2] = 3.625$

2nd iteration:

$u_1^{(2)} = 1.625$; $u_2^{(2)} = 2.8125$; $u_4^{(2)} = 3.9063$

3rd iteration:

$u_1^{(3)} = 1.9063$; $u_2^{(3)} = 2.9532$; $u_4^{(3)} = 3.9766$

4th iteration:

$u_1^{(4)} = 1.9766$; $u_2^{(4)} = 2.9883$; $u_4^{(4)} = 3.9942$

5th iteration:

$u_1^{(5)} = 1.9942$; $u_2^{(5)} = 2.9971$; $u_4^{(5)} = 3.9986$

6th iteration:

$u_1^{(6)} = 1.9986$; $u_2^{(6)} = 2.9993$; $u_4^{(6)} = 3.9997$

7th iteration:

$u_1^{(7)} = 1.9997$; $u_2^{(7)} = 2.9999$; $u_4^{(7)} = 4$

8th iteration:

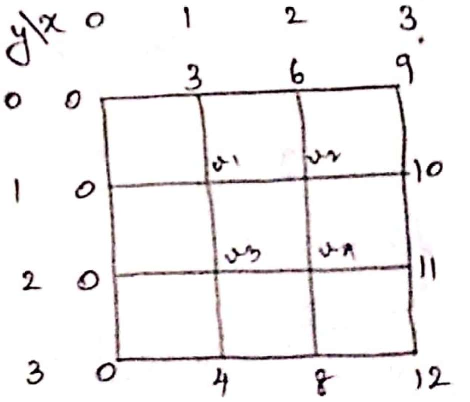
$u_1^{(8)} = 2$; $u_2^{(8)} = 3$; $u_4^{(8)} = 4$

9th iteration:

$u_1^{(9)} = 2$; $u_2^{(9)} = 3$; $u_4^{(9)} = 4$

Hence $u_1 = 2$, $u_2 = 3$, $u_3 = 3$, $u_4 = 4$

⑤ By iteration method, solve the Laplace eqn. $u_{xx} + u_{yy} = 0$ over the square region satisfying the boundary conditions. $u(0,y) = 0$, $0 \le y \le 3$; $u(3,y) = 9+y$, $0 \le y \le 3$; $u(x,0) = 3x$, $0 \le x \le 3$; $u(x,3) = 4x$, $0 \le x \le 3$.



Sol: Given $u_{xx} + u_{yy} = 0$, $u(0,y) = 0$, $0 \le x \le 3$; $u(3,y) = 9+y$, $0 \le y \le 3$; $u(x,0) = 3x$, $0 \le x \le 3$; $u(x,3) = 4x$, $0 \le x \le 3$. Let the internal grid pts be u_1, u_2, u_3, u_4.

Rough values: Assume $u_4 = 0$

$u_1 = \frac{1}{4}[0+6+0+u_4] = 1.5$ [DFPF]

$u_2 = \frac{1}{4}[16+u_1+u_4] = 4.375$ [SFPF]

$u_3 = \frac{1}{4}[0+4+u_1+u_4] = 1.375$ [SFPF]

$u_4 = \frac{1}{4}[19+u_2+u_3] = 6.1875$ [SFPF]

1st iteration:

$u_1^{(1)} = \frac{1}{4}[0+3+u_2+u_3] = 2.1875$

$u_2^{(1)} = \frac{1}{4}[16+u_1+u_4] = 6.0938$

$u_3^{(1)} = \frac{1}{4}[4+u_1+u_4] = 3.0938$

$u_4^{(1)} = \frac{1}{4}[19+u_2+u_3] = 7.0469$

2^{nd} iteration:

$u_1^{(2)} = 3.0469$; $u_2^{(2)} = 6.5235$; $u_3^{(2)} = 3.5235$; $u_4^{(2)} = 7.2618$

3^{rd} iteration:

$u_1^{(3)} = 3.2618$; $u_2^{(3)} = 6.6309$; $u_3^{(3)} = 3.6309$; $u_4^{(3)} = 7.3155$

4^{th} iteration:

$u_1^{(4)} = 3.3155$; $u_2^{(4)} = 6.6578$; $u_3^{(4)} = 3.6578$; $u_4^{(4)} = 7.3289$

5^{th} iteration:

$u_1^{(5)} = 3.3289$; $u_2^{(5)} = 6.6645$; $u_3^{(5)} = 3.6645$; $u_4^{(5)} = 7.3323$

6^{th} iteration:

$u_1^{(6)} = 3.3323$; $u_2^{(6)} = 6.6662$; $u_3^{(6)} = 3.6662$; $u_4^{(6)} = 7.3331$

7^{th} iteration:

$u_1^{(7)} = 3.3331$; $u_2^{(7)} = 6.6666$; $u_3^{(7)} = 3.6666$; $u_4^{(7)} = 7.3333$

Hence $u_1 = 3.33$, $u_2 = 6.67$, $u_3 = 3.67$, $u_4 = 7.33$

Two dimensional Poisson eqn.:

Any eqn. of the form $\nabla^2 u = f(x,y)$ (ie) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$ — ① is called as Poisson eqn. where $f(x,y)$ is a fun. of x & y only.

WKT $u_{xx} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}$, $u_{yy} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2}$

Take $x = ih$, $y = jk = jh$

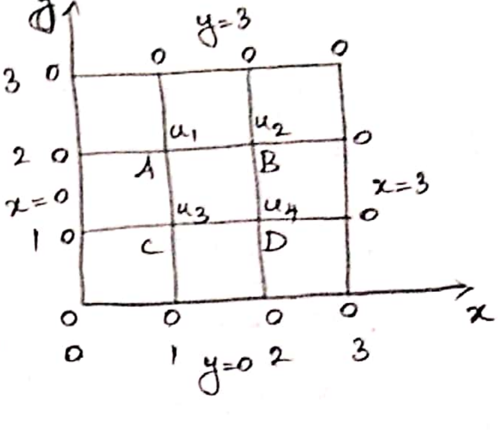
$\therefore ① \Rightarrow u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = h^2 f(ih, jh)$ — ②

By applying the above formula at each mesh pt, we get a system of linear eqn. in the pivotal values i,j.

Problems:

① Solve the Poisson eqn. $\nabla^2 u = -10(x^2+y^2+10)$ over the square with sides $x=0, y=0, x=3, y=3$ with $u=0$ on the boundary, taking $h=1$.

Sol:



Let the values of u at the four mesh pts A, B, C & D be u_1, u_2, u_3, u_4 respectively. The differential eqn. is $\nabla^2 u = -10(x^2+y^2+10)$ —①

Here $h=1$. Put $x=ih=i$, $y=jh=j$

$u_{i-1,j}+u_{i+1,j}+u_{i,j-1}+u_{i,j+1}-4u_{i,j} = -10(i^2+j^2+10)$ —②

At A $i=1, j=2$

$0+u_2+u_3+0-4u_1 = -150 \Rightarrow -4u_1+u_2+u_3 = -150$ —③

At B $i=2, j=2$

$u_1+0+u_4+0-4u_2 = -180 \Rightarrow u_1-4u_2+u_4 = -180$ —④

At C $i=1, j=1$

$0+u_4+0+u_1-4u_3 = -120 \Rightarrow u_1-4u_3+u_4 = -120$ —⑤

At D $i=2, j=1$

$u_3+0+0+u_2-4u_4 = -150 \Rightarrow u_2+u_3-4u_4 = -150$ —⑥

From ③ & ⑥ we get, $u_1 = u_4$

$\therefore$ ④ $\Rightarrow 2u_1 - 4u_2 = -180$ —⑦

$\therefore$ ⑤ $\Rightarrow 2u_1 - 4u_3 = -120$ —⑧

$\cancel{⑦\times 2} \Rightarrow \cancel{4u_1 - 8u_2 = -360}$

$\cancel{-4u_1 + u_2 + u_3 = -150}$ —$\cancel{③}$

$\cancel{-7u_2 + u_3 = -510}$

$⑧ \times \cancel{4} \Rightarrow \cancel{8}$

$③\times 4 \Rightarrow -16u_1 + 4u_2 + 4u_3 = -600$

$2u_1 - 4u_2 = -180$ —⑦

$-14u_1 + 4u_3 = -780$

$2u_1 - 4u_3 = -120$ —⑧

$-12u_1 = -900$

$\boxed{u_1 = 75}$

Subs. u_1 in ⑧, $4u_3 = 270 \Rightarrow \boxed{u_3 = 67.5}$

Subs. u_1 in ⑦, $4u_2 = 330 \Rightarrow \boxed{u_2 = 82.5}$

Hence $u_1 = 75$, $u_2 = 82.5$, $u_3 = 67.5$, $u_4 = 75$

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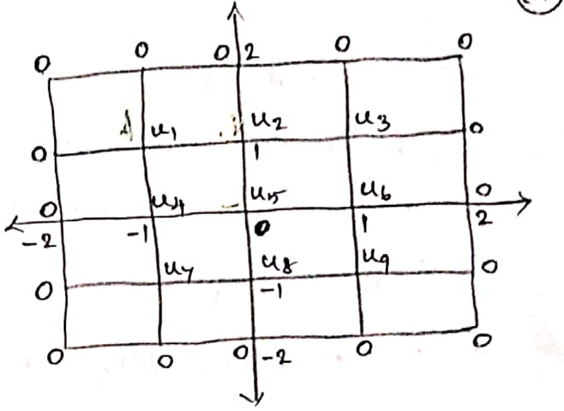
② Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 8x^2y^2$ in the square mesh given $u=0$ on the 4 boundaries dividing the square into 16 subsquares of length 1 unit.

Sol: Here $h=1$, $u=0$



Let $u_1, u_2, \ldots, u_9$ be the values of u at the interior grid pts.

The given Poisson PDE $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 8x^2y^2$ is symmetrical about x & y axes & also about the line $y = x$.

Hence we have $u_1 = u_3 = u_7 = u_9$ & $u_2 = u_4 = u_6 = u_8$

Hence we have to find u_1, u_2, u_5 only. Here $\nabla^2 u = 8x^2y^2$. Put $x = ih = i$, $y = jh = j$

$u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = 8i^2j^2$

At u_1 $i = -1$, $j = +1$

$0 + 0 + u_2 + u_4 - 4u_1 = 8 \Rightarrow -4u_1 + 2u_2 = 8 \Rightarrow -2u_1 + u_2 = 4$ ——①

At u_2 $i = 0$, $j = 1$

$0 + u_1 + u_5 + u_3 - 4u_2 = 0 \Rightarrow 2u_1 - 4u_2 + u_5 = 0$ ——②

At u_5 $i = 0$, $j = 0$

$u_2 + u_4 + u_8 + u_6 - 4u_5 = 0 \Rightarrow 4u_2 - 4u_5 = 0 \Rightarrow u_2 - u_5 = 0$ ——③

① + ② $\Rightarrow$ $-3u_2 + u_5 = 4$

$u_2 - u_5 = 0$ ——③

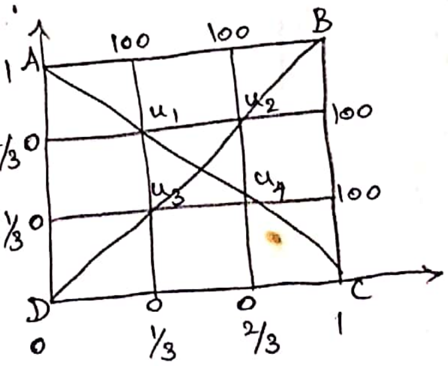
$-2u_2 \quad = 4 \Rightarrow \boxed{u_2 = -2}$

Subs. u_2 in ①, $2u_1 = -2 - 4 \Rightarrow \boxed{u_1 = -3}$

Subs. u_2 in ③, $\boxed{u_5 = -2}$

Hence $u_1 = u_3 = u_7 = u_9 = -3$, $u_2 = u_4 = u_6 = u_8 = -2$, $u_5 = -2$

③ Solve the Poisson eqn. $u_{xx} + u_{yy} = -81xy$, $0 < x < 1$, $0 < y < 1$ given that $u(0,y) = 0$, $u(x,0) = 0$, $u(1,y) = 100$, $u(x,1) = 100$ & $h = 1/3$.



Sol: Given $u_{xx} + u_{yy} = -81xy$, $0 < x < 1$, $0 < y < 1$

$u(0,y) = 0$, $u(x,0) = 0$, $u(1,y) = 100$, $u(x,1) = 100$.

$h = 1/3$, $x = ih = i/3 \Rightarrow 3x = i$, $y = jh = j/3 \Rightarrow 3y = j$

From the figure, by symmetry we get $u_1 = u_4$

Hence we have to find u_1, u_2 & u_3 only. $u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = \frac{-81ij}{81}$

$\Rightarrow u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = -ij$

At u_1 $i = 1$, $j = 2$

$100 + 0 + u_3 + u_2 - 4u_1 = -2 \Rightarrow -4u_1 + u_2 + u_3 = -102$ ——①

At u_2 $i=2, j=2$

$100+u_1+u_4+100-4u_2=-4$

$2u_1-4u_2=-204 \Rightarrow u_1-2u_2=-102$ — ②

At u_3 $i=1, j=1$

$u_1+0+0+u_4-4u_3=-1 \Rightarrow 2u_1-4u_3=-1$ — ③

① × 2 $\Rightarrow -8u_1+2u_2+2u_3=-204$

$u_1-2u_2 \quad =-102$

$-7u_1 \quad +2u_3=-306$ — ④

④ × 2 $\Rightarrow -14u_1+4u_3=-612$

$2u_1-4u_3=-1$ — ③

$-12u_1 \quad =-613$

$$\boxed{u_1=51.0833}$$

Subst. u_1 in ②, $2u_2=153.0833 \Rightarrow \boxed{u_2=76.5417}$

Subst. u_1 in ③, $4u_3=103.1666 \Rightarrow \boxed{u_3=25.7917}$

Hence $u_1=51.0833$, $u_2=76.5417$, $u_3=25.7917$, $u_4=51.0833$

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